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Robert's Recursive Frequency Filter: A Reexamination

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With 4 Figures

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Summary

The recursive filter designed by Robert (1966) has been investigated in detail. The filter is found to have a dual impact on the wave on which it acts in the sense that it changes the phase and amplitude of the wave unlike a nonrecursive filter which only changes the wave amplitude. The dual nature of the filter's effect is attributed primarily to its recursive nature. Procedures for choosing the correct value of the smoothing element of the filter for achieving some desired postfiltration results are also discussed.

1. Introduction

The recursive frequency filter devised by Robert (1966) has been widely used by many researchers for controlling the temporal oscillations of the solutions arising during the time integration procedures of their numerical models. Asselin (1972) has studied the implications of this frequency filter using the leap-frog scheme in the integration process. We intend to study the effects of this filter with the Adams-Bashforth scheme in a simple advective equation as well as some intrinsic details of this filter. The amplification factor for the "Adams-Bashforth scheme—Robert's filter" combination should have contributions from the finite difference scheme as well as from the filter. By an analysis of the filter we shall see how a proper selection of the smoothing element of the filter affects the achievement of some of the preconceived postfiltered contributions of the filter alone.

2. Filter's Intrinsic Details

The filter as devised by Robert (1966) is:

$$\varphi^*(t) = \varphi(t) + F(\varphi(t+\Delta t) + \varphi^*(t-\Delta t) - 2\varphi(t)) \quad (1)$$

where φ represents the unfiltered physical quantity undergoing filter application, F is the smoothing element of the filter, φ^* denotes the physical quantity after filtering, t stands for time and Δt for the temporal increment.

A recursive procedure is defined as a procedure which uses itself in evaluating itself. The filter defined in (1) is recursive as the evaluation of the filtered value at time t needs the filtered value at the earlier time step. The occurrence of the filtered value viz $\varphi^*(t-\Delta t)$ on the RHS of eq. (1) makes the filter recursive. Let R be the Response Function of the filter.

$$\varphi^*(t) = R\varphi(t) \quad (2)$$

Equation (1) in conjunction with (2) yields:

$$R\varphi(t) = \varphi(t) + F(\varphi(t+\Delta t) + \varphi^*(t-\Delta t) - 2\varphi(t)) \quad (3)$$

Assume that φ represents a wave of unit amplitude and angular frequency ω.

$$\varphi(t) = e^{i\omega t} \tag{4}$$

where angular frequency $\omega = \frac{2\pi}{T}$, T being the period of the wave.

Substitution of (4) in (3) gives:

$$R e^{i\omega t} = e^{i\omega t} + F(e^{i\omega(t+\Delta t)} + R\varphi(t-\Delta t) - 2e^{i\omega t}) \tag{5}$$

where $\varphi^*(t-\Delta t)$, of RHS of (3), is replaced by $\varphi(t-\Delta t)$ as per eq. (2).

Equation (5) implies:

$$R e^{i\omega t} = e^{i\omega t} + F(e^{i\omega(t+\Delta t)} + R\, e^{i\omega(t-\Delta t)} - 2e^{i\omega t})$$

$$R = 1 + F(e^{i\omega\Delta t} + R\, e^{-i\omega\Delta t} - 2)$$

$$R(1 - F e^{-i\omega\Delta t}) = 1 - 2F + F e^{i\omega\Delta t}$$

$$R(1 - F\cos\omega\Delta t + iF\sin\omega\Delta t) = 1 - 2F + F\cos\omega\Delta t + iF\sin\omega\Delta t$$

$$R(1 + F^2 - 2F\cos\omega\Delta t) = (1 - 2F + F\cos\omega\Delta t)(1 - F\cos\omega\Delta t) + F^2\sin^2\omega\Delta t + i\{F\sin\omega\Delta t(1 - F\cos\omega\Delta t) - F\sin\omega\Delta t(1 - 2F + F\cos\omega\Delta t)\}$$

i.e.

$$R = \frac{1 - 2F + 2F^2\cos\omega\Delta t - F^2\cos 2\omega\Delta t}{1 + F^2 - 2F\cos\omega\Delta t} + i\frac{2F^2\sin\omega\Delta t(1 - \cos\omega\Delta t)}{1 + F^2 - 2F\cos\omega\Delta t} \tag{6}$$

Equation (6) shows that the filter's response function R is complex and is a function of angular frequency (ω) of the wave temporal grid size Δt and the smoothing element F. As the angular frequency ω of a wave is related to its period T by $\omega = \frac{2\pi}{T}$, so the complex response function R is implicitly a function of the period also. We again express the period of the wave in terms of the temporal increment Δt as $T = n\Delta t$ where the multiplicity factor n will take different values for different periods.

So, the functional dependence of the complex response function R will be on the smoothing element F, the period of the wave T and the multiplicity factor n, introduced earlier. For the purposes of interpretation, a wave will be referred to by the corresponding value of the multiplicity factor n. Just as in spatial filters, in which a particular wave's wavelength is specified in terms of its multiplicity to the grid length (e.g. 2 grid wave, 3 grid wave etc.), a $2\Delta t$ wave in this temporal filter will be the wave whose period is twice the temporal increment or temporal grid. For the purposes of interpretation, the temporal increment Δt will more or less be identified as temporal grid and the waves will be referred to as the waves with period twice the grid ($2\Delta t$), thrice the grid ($3\Delta t$) and so forth. Eq. (6) can be re-expressed as:

$$R = R_{real} + i\,R_{imag} = |R|\,e^{i\theta} \tag{7}$$

where R_{real} is the real part of R; R_{imag} is the imaginary part of R; $|R|$ is the positive square root of $(R^2_{real} + R^2_{imag})$; and

$$\tan\theta = \frac{R_{imag}}{R_{real}}.$$

Comparison of (6) and (7) gives:

$$R_{real} = |R|\cos\theta = \frac{1 - 2F + 2F^2\cos\omega\Delta t - F^2\cos 2\omega\Delta t}{1 + F^2 - 2F\cos\omega\Delta t}$$

and

$$R_{imag} = |R|\sin\theta = \frac{2F^2(1 - \cos\omega\Delta t)\sin\omega\Delta t}{1 - F^2 - 2F\cos\omega\Delta t}$$

$$|R| = \{1 - 4F + 6F^2 - 4F^3 + 5F^4 + (4F^2 - 8F^3 - 4F^4)\cos\omega\Delta t - (4F^2 - 8F^3)\cos^2\omega\Delta t\}^{\frac{1}{2}} (1 + F^2 - 2F\cos\omega\Delta t)^{-1} \tag{8 a}$$

and

$$\tan\theta = \left(\frac{2F^2(1 - \cos\omega\Delta t)\sin\omega\Delta t}{1 + F^2 - 2F\cos\omega\Delta t}\right) \left(\frac{1 - 2F + 2F^2\cos\omega\Delta t - F^2\cos 2\omega\Delta t}{1 + F^2 - 2F\cos\omega\Delta t}\right)^{-1} \tag{8 b}$$

Equation (2) in conjunction with eq. (7) yields:

$$\varphi^*(t) = R\varphi(t) = |R|\,e^{i\theta}\,\varphi(t) \tag{9}$$

and as $\varphi(t)$ is assumed to be a wave of unit amplitude as

$\varphi(t) = e^{i\omega\Delta t}$, so eq. (9) becomes:

$$\varphi^*(t) = |R|\,e^{i(\omega t + \theta)} \tag{10}$$

This equation represents the postfiltering structure of an arbitrary wave of unit amplitude. This equation shows that a) the amplitude of the wave has changed from one unit to $|R|$ units and b) the phase of the wave changes by θ. From this we get the percentage change, fall or enhancement, introduced by the filter in the amplitude of the wave as $(1-|R|)\times 100$.

From eq. (8 a) and (8 b) we find that $|R|=1$ and $\tan\theta=0$; i.e. $|R|=1$ and $\theta=0$; whenever $F=0$. This implication of (8 a) and (8 b) is correct because according to the definition proposed by Robert (1966) and given in eq. (1), the above substitution viz $F=0$ is tantamount to no filtration, as eq. (1) with $F=0$ becomes:

$$\varphi^*(t)=\varphi(t).$$

2.1 Some Special Cases

$|R|$ and θ are the parameters defining the post-filtering effects of the filter on a wave of unit amplitude and given phase. We shall now see how the eq. (8 a) and (8 b) modify for some particular preconceived postfiltering values of $|R|$ and θ thereby leading to the computations of the appropriate values of the smoothing element F. Filter's contribution obtained by using any of these values of F will be the same as the one which was used in deriving this particular value of F.

Case 1

Firstly we consider the effect of the filter for any arbitrary value of the smoothing element F on the waves of angular frequency ω satisfying $\cos\omega\Delta t = 1$.

Now with $\cos\omega\Delta t=1$, $|R|$ from eq. (8 a) becomes:

$$|R|=(1-4F+6F^2-4F^3+F^4)^{\frac{1}{2}}(1+F^2-2F)^{-1}$$
$$=((1+F^2-2F)^2)^{\frac{1}{2}}(1+F^2-2F)^{-1}$$

i.e.

$$|R|=1$$

and from (8 b) we get

$$\tan\theta=0\Rightarrow\theta=p\pi\;\forall\,p=0,\pm1,\pm2,\ldots$$

The number of different grid waves implied by $\cos\omega\Delta t=1$ are computed as under.

$$\cos\omega\Delta t=1$$

$$\Rightarrow\frac{2\pi}{n\Delta t}\Delta t=2m\pi\;\forall\; m=1,2,3,\ldots$$

i.e.

$$n=\frac{1}{m}\;\forall\; m=1,2,3,\ldots$$

i.e.

$$n=1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\ldots$$

$n=1$ is the only physically acceptable value of the multiplicity factor n. From this it can be concluded that the amplitude of one grid wave will always be invariant under this filter, for any value of F except $F=1$, while its phase will change by an integral multiple of π. Value $F=1$ is excluded as $|R|$ becomes indeterminate $\frac{0}{0}$ whenever $F=1$ and $\cos\omega\Delta t=1$.

Case 2

Now, we consider those angular frequencies for which the absolute value of the complex response function vanishes for any arbitrary value of the smoothing element F i.e. we will be looking for those values of the multiplicity factor n for which $|R|=0$ for some F.

From eq. (8 a) we have:

$$|R|=\{1-4F+6F^2-4F^3+5F^4+(4F^2-8F^3-4F^4)\cos\omega\Delta t-(4F^2-8F^3)\cos^2\omega\Delta t\}^{\frac{1}{2}}(1+F^2-2F\cos\omega\Delta t)^{-1}$$

Now

$$|R|=0$$

$$\Rightarrow(4F^2-8F^3)\cos^2\omega\Delta t-(4F^2-8F^3-4F^4)\cos\omega\Delta t-(1-4F+6F^2-4F^3+5F^4)=0 \quad (10)$$

This is a quadratic equation in $\cos\omega\Delta t$ and its solution is:

$$\cos\omega\Delta t=(4F^2-8F^3-4F^4)\pm\{16F^2(F-1)^6\}^{\frac{1}{2}}(2\{4F^2-8F^3\})^{-1}$$

or

$$\cos\omega\Delta t=\cos\frac{2\pi}{m}$$
$$=\{4F^2-8F^3-4F^4\pm4F(F-1)^3\}(8(1-2F)F^2)^{-1} \quad (11)$$

In this particular case we will consider two sub-cases viz. one for each of $\pm$ sign appearing in the numerator of eq. (11).

Subcase (i)

With positive sign eq. (11) yields:

$$\cos\omega\Delta t=(4F^2-8F^3-4F^4+4F(F-1)^3)(8F^2-16F^3)^{-1}$$

i.e.

$$\cos\omega\Delta t=(1-4F+5F^2)(4F^2-2F)^{-1}$$

Subcase (ii)

With negative sign eq. (11) becomes:

$$\cos\omega\,\Delta\,t = (4F^2 - 8F^3 - 4F^4 - 4F(F-1)^3)\,.\,(8F^2 - 16F^3)^{-1}$$

i.e.

$$\cos\omega\,\Delta\,t = \frac{1}{2}(1+F^2)\,F^{-1}$$

$$\Rightarrow F^2 - 2F\cos\omega\,\Delta\,t + 1 = 0$$

i.e. the choice of a negative sign reduces $|R|$ to the indeterminate form $\frac{0}{0}$ which is unphysical and unacceptable. We find that only a positive sign leads to physically acceptable conclusions. On retaining only the positive sign eq. (11) reduces to the form.

$$\cos\omega\,\Delta\,t = \cos\frac{2\pi}{n}$$

$$= (1 - 4F + 5F^2)(4F^2 - 2F)^{-1} \qquad (12)$$

Equation (12) can be solved either for n for some given value of F or for F for some given value of n. Former solution will give us that value of the smoothing element F for which the corresponding filter will completely filter that grid wave which was used in computing this value of F. Latter solution will give us the grid waves which will be completely filtered by the filter having a particularly selected value of F. Assume that we wish to evaluate the value of the smoothing element F for which the corresponding filter will filter the waves of twice-grid period, i.e. we intend to find that value of F for which the wave of period $T = 2\,\Delta\,t$ will have $|R| = 0$. Now for the waves of this period

$$\cos(\omega\,\Delta\,t) = \cos\left(\frac{2\pi}{T}\Delta\,t\right) = \cos\left(\frac{2\pi}{2\,\Delta\,t}\Delta\,t\right) = \cos\pi = -1$$

From subcase (i) we get:

$$-1 = (5F^2 - 4F + 1)(2F(2F-1))^{-1}$$

$$5F^2 - 4F + 1 + 2F(2F-1) = 0$$

$$(3F-1)^2 = 0 \quad \text{i.e. } F = 1/3$$

i.e. frequency filter with $F = 1/3$ will give $|R| = 0$ for 2-grid waves. Conversely it can be seen which grid waves are implied by eq. (12) when a filter with $F = 1/3$ is used.

$$\cos\omega\,\Delta\,t = [(5F^2 - 4F + 1)(4F^2 - 2F)^{-1}]_{F=1/3}$$

$$= \left(\frac{2}{9}\right)\left(-\frac{2}{9}\right)^{-1} = -1$$

i.e.

$$\omega\,\Delta\,t = \frac{2\pi}{n} = \pi \Rightarrow n = 2$$

2-grid waves will be completely filtered by a filter with $F = 1/3$ while earlier we found that for filtering 2-grid waves, the filter should have $F = 1/3$. For filtering thrice-grid ($T = 3\,\Delta\,t$) period waves, the smoothing element F becomes complex and it works out to be $F = \frac{5 \pm i\sqrt{3}}{14}$. In this case $(3\,\Delta\,t)$, we find two values of F giving the same effect. F has two values primarily because the complex roots of a quadratic equation of real coefficients always coexist as a conjugate pair.

Case 3

In this case, we intend to find those values of the smoothing element F for which the absolute value of the complex response functions, $|R|$, as defined in (8 a), never exceeds 1, i.e. we will find those F for which $|R| \leqslant 1$ for a wave of given multiplicity factor n.

Equation (8 a) in conjunction with the condition $|R| \leqslant 1$ yields:

$$\{1 - 4F + 6F^2 - 4F^3 + 5F^4 + (4F^2 - 8F^3 - 4F^4)\cos\omega\,\Delta\,t - (4F^2 - 8F^3)\cos^2\omega\,\Delta\,t\}^{\frac{1}{2}} \leqslant 1 + F^2 - 2F\cos\omega\,\Delta\,t$$

This inquality on simplification leads to the following inequality.

$$8F^2(F-1)\left(\frac{F^2+1}{2F} - \cos\omega\,\Delta\,t\right) \leqslant 0 \qquad (13)$$

In eq. (13) term $\cos\omega\,\Delta\,t\left(= \cos\frac{2\pi}{n}\right)$ is known as we are looking for those values of F which $|R| \leqslant 1$ for a wave of given n. We shall see how this inequality leads to the selection of those values of F which are appropriate for effecting $|R| \leqslant 1$ for some particular grid wave. Firstly we consider 2-grid waves.

For $n = 2$ the inequality of eq. (13) becomes

$$8F(F-1)(F+1)^2 \leqslant 0$$

or

$F(F-1) \leqslant 0$ as $(F+1)^2$ being a squared quantity is always positive. i.e. F must be in the range $0 \leqslant F \leqslant 1$ for not allowing the postfiltering amplitude of 2-grid waves to exceed unity.

Secondly we consider one grid waves and for $n = 1$ inequality of eq. (13) reduces to an equality $(0 = 0)$ thereby implying that for one grid waves $|R| = 1$ i.e. the amplitude of one grid waves will always be invariant under any filter. This supports the conclusion drawn earlier from case 1 wherein it was shown that $|R| = 1$ for one grid waves for a frequency filter of any F. However for an arbitrary grid wave, F must be chosen in conformity with the inequality given in eq. (13).

Case 4

Under this case we study the behavior of the filter when the smoothing element $F = 1$. Substitution of $F = 1$ in eq. (8 a) gives $(|R|)_{F=1} = 1$ i.e. the amplitudes of waves of any angular frequency will be invariant under this filter. The impact of this filter on the phase of an arbitrary wave is obtained from the transformation of eq. (8 b) under $F = 1$.

i.e.

$$\tan\theta = \left(\frac{2F^2(1-\cos\omega\Delta t)\sin\omega\Delta t}{1+F^2-2F\cos\omega\Delta t}\right)$$
$$\left(\frac{1-2F+2F^2\cos\omega\Delta t - F^2\cos 2\omega\Delta t}{1+F^2-2F\cos\omega\Delta t}\right)^{-1}$$
$$(\tan\theta)_{F=1} = \tan\omega\Delta t$$

i.e.

$$\theta = \omega\Delta t$$

Here θ is a measure of the phase change introduced by the filter. Now

$$\theta = \omega\Delta t = \frac{2\pi}{T}\Delta t = \frac{2\pi}{n\Delta t}\Delta t = \frac{2\pi}{n}$$

i.e.

$$\theta = \frac{2\pi}{n}$$

So, the complex response function, for $F = 1$ becomes

$$R = |R|e^{i\theta} = e^{\frac{i2\pi}{n}} \quad \text{as} |R| = 1$$

This leads to the conclusion that the amplitude of wave of any multiplicity factor n will be invariant under this filter though its phase will change by an amount $\frac{2\pi}{n}$, i.e. the filter with $F = 1$ changes only the phase of waves, leaving their amplitudes unaffected.

Case 5

Contrary to the case 3 wherein we evaluated those values of the smoothing (filter) element F for which the filter damps all waves, we shall now see for what values of F the filter amplifies the amplitudes of all waves i.e. for what values of F, $|R| \geqslant 1$ for all ω. Equation (8 a) in conjunction with $|R| \geqslant 1$ gives an inequality for an arbitrary n as

$$8F^2(F-1)\left(\frac{F^2+1}{2F} - \cos\frac{2\pi}{n}\right) \geqslant 0 \qquad (14)$$

In particular this inequality reduces to equality $(0 = 0)$ for one grid waves $\left(\cos\frac{2\pi}{n} = -1\right)$ and to the form $F(F-1) \geqslant 0$ for 2-grid waves $\left(\cos\frac{2\pi}{n} = -1;\ F(F-1) \geqslant 0\right)$ implies that F must in the range $0 > F > 1$ for the filter to have amplification effect on the amplitudes of 2-grid waves. However for an arbitrary grid wave, F must be chosen in accordance to the inequality given in eq. (14).

3. Filter with the Adams-Bashforth Scheme

We shall now study the effect of this filter when used in conjunction with the Adams-Bashforth finite different scheme in a simple advective equation of the form

$$\frac{d\varphi}{dt} = i\omega\varphi \qquad (15)$$

which by using the Adams-Bashforth difference scheme, can be written as

$$\varphi_{t+\Delta t} = \varphi_t + \frac{i\omega}{2}(3\varphi_t - \varphi^*_{t-\Delta t}) \qquad (16)$$

Scheme has a computational mode besides a physical mode as it is a 3-level scheme. The scheme tends to damp the computational mode and amplify the physical mode. In eq. (16) smoother function at $t-\Delta t$ is obtained from eq. (1) using values at $t-2\Delta t$, $t-\Delta t$ and t. We now define "Amplification Factor X" by expression

$$\varphi_{t+\Delta t} = X\varphi_t \tag{17}$$

For simplicity, we shall henceforth assume the temporal grid Δt to be of unit time i.e. $\Delta t = 1$. Combining eq. (1), (16) and (17) we find that

$$X_{1,2} = \frac{(F+1)+\dfrac{i\omega}{2}(3-F)}{2} \pm \frac{\left\{(F-1)^2-\dfrac{\omega^2}{4}(3-F)^2-i\omega(F^2-1)\right\}^{\frac{1}{2}}}{2} \tag{18}$$

The negative sign associated with the radicand corresponds to the computational mode. The amplification factor X represents the response of the "filter-difference scheme" combination on a wave of given frequency. Substitution of $F = 0$ in eq. (18) leads to

$$X_{sc} = \frac{1}{2} + \frac{3i\omega}{4} \pm \frac{1}{2}\left(1-\frac{9\omega^2}{4}+i\omega\right)^{\frac{1}{2}} \tag{19}$$

viz. the amplification factor for the Adams-Bashforth scheme alone as $F = 0$ implies no filter usage. The amplification factor X, as defined in eq. (18), will have a contribution from the filter as well as from the Adams-Bashforth scheme. Filter's contribution is represented by its response function, eq. (6), and that of scheme's is represented by its amplification factor given in eq. (19). During a time integration procedure, the effect of filter-scheme combination goes on accumulating and what we get after n steps of integration is the cumulative effect of this combination. We shall consider the cumulative effect of this frequency filter scheme combination for one particular case viz. for $F = 3$. The choice of this particular value of F is motivated by the wish for simplicity in eq. (18). For $F = 3$ we get from eq. (18)

$$X^{(3)} = 2 \pm 0.707\{(1+4\omega^2)^{\frac{1}{2}}+1\}^{\frac{1}{2}} \pm 0.707i\{(1+4\omega^2)^{\frac{1}{2}}-1\}^{\frac{1}{2}}$$

and the effect after n time-steps is given as $X^{(3)n}$ i.e.

$$X^{(3)n} = |X^{(3)}|^n e^{in\theta} \tag{20}$$

where

$$|X^{(3)}| = [4+(1+4\omega^2)^{\frac{1}{2}} + 2.828\{(1+4\omega^2)^{\frac{1}{2}}+1\}^{\frac{1}{2}}]^{\frac{1}{2}} \tag{20 a}$$

$$\tan\theta = \{(1+4\omega^2)^{\frac{1}{2}}-1\}^{\frac{1}{2}} [2.828 \pm \{(1+4\omega^2)^{\frac{1}{2}}+1\}^{\frac{1}{2}}]^{-1} \tag{20 b}$$

Contributions of the filter and the difference scheme to the cumulative results of eq. (20) will also be cumulative. The cumulative contribution of the frequency filter is R^n where R is obtained from eq. (8 a), (8 b) after substituting $F = 3$. i.e.

$$R^n = |R|^n e^{in\theta} \tag{21}$$

where

$$|R| = (85-126\cos\omega+45\cos^2\omega)(5-3\cos\omega)^{-1} \tag{21 a}$$

and

$$\tan\theta = \left(\frac{9\sin\omega(1-\cos\omega)}{5-3\cos\omega}\right)\left(\frac{2+9\cos\omega-9\cos^2\omega}{5-3\cos\omega}\right)^{-1} \tag{21 b}$$

and the cumulative effect arising from the Adams-Bashforth scheme is X^n_{sc} with X_{sc} defined as in eq. (19).

$$X^n_{sc} = |X_{sc}|^n e^{in\theta} \tag{22}$$

where

$$|X_{sc}| = \frac{1}{4}\left[4+9\omega^2+4\left(1+\frac{81\omega^4}{16}-\frac{7\omega^2}{2}\right)^{\frac{1}{2}} + 5.656\left\{\left(1+\frac{81\omega^4}{16}-\frac{7\omega^2}{2}\right)^{\frac{1}{2}} + 1-\frac{9\omega^2}{4}\right\}^{\frac{1}{2}} + 8.484\left\{\left(1+\frac{81\omega^4}{16}-\frac{7\omega^2}{2}\right)^{\frac{1}{2}} - 1+\frac{9\omega^2}{4}\right\}^{\frac{1}{2}}\right]^{\frac{1}{2}} \tag{22 a}$$

and

$$\tan\theta = \frac{2.121+\left(\left(1+\dfrac{81\omega^4}{16}-\dfrac{7\omega^2}{2}\right)^{\frac{1}{2}}-1+\dfrac{9\omega^2}{4}\right)^{\frac{1}{2}}}{1.414+\left(\left(1+\dfrac{81\omega^4}{16}-\dfrac{7\omega^2}{2}\right)^{\frac{1}{2}}+1-\dfrac{9\omega^2}{4}\right)^{\frac{1}{2}}} \tag{22 b}$$

The amplification factor for the case when leap frog scheme is used with the frequency filter in the frequency integration process is given as

$$X = F + i\omega \pm ((F-1)^2 - \omega^2)^{\frac{1}{2}} \tag{23}$$

The leap frog scheme, being a 3-level scheme, has a computational mode besides the physical mode and the negative sign associated with the radicand corresponds to the computational mode. The cumulative effect of n steps of temporal integration is given by X^n where:

$$X^n = |X|^n e^{in\theta} \tag{23 a}$$

In this case X will take two different forms depending on the sign of the quantity $(F-1)^2 - \omega^2$. When this quantity is positive we get

$$X^{(+)n} = |X^{(+)}|^n e^{in\theta^{(+)}} \tag{24}$$

where

$$|X^{(+)}| = (2F^2 - 2F + 1 \pm 2F (F^2 - \omega^2 - 2F + 1)^{\frac{1}{2}})^{\frac{1}{2}} \tag{24 a}$$

and

$$\tan\theta^{(+)} = \omega (F \pm (F^2 - \omega^2 - 2F + 1)^{\frac{1}{2}})^{-1} \tag{24 b}$$

and when $(F-1)^2 - \omega^2$ is negative we get

$$X^{(-)n} = |X^{(-)}|^n e^{in\theta^{(-)}} \tag{25}$$

where

$$|X^{(-)}| = (2\omega^2 + 2F - 1 \pm 2\omega (\omega^2 - F^2 + 2F - 1)^{\frac{1}{2}})^{\frac{1}{2}} \tag{25 a}$$

and

$$\tan\theta^{(-)} = (\omega \pm (\omega^2 - F^2 + 2F - 1)^{\frac{1}{2}}) F^{-1} \tag{25 b}$$

Like in the Adams-Bashforth scheme case, the amplification factor for the leap frog scheme is obtained by substituting $F = 0$ in eq. (23) and is given as

$$X_{LF} = i\omega \pm (1 - \omega^2)^{\frac{1}{2}} \tag{26}$$

Fig. 1 a gives the profiles of $|R|$ for certain values of the filter parameter F with $\Delta t = 1$ and Table 1 shows the maximum postfiltering amplitude obtained by applying filter on waves in the frequency range (0.1, 1.2) for 16 different combinations of $(\Delta t, F)$ with $\Delta t = 1$, ¾, ½, ¼ and F = ⅛, ¼, ⅓, ½. It is seen from Table 1 that the maximum postfiltering amplitude 1) increases as Δt decreases for fixed F and 2) increases as F decreases for fixed Δt. Fig. 1 b shows the phase changes (in degrees) introduced by the application of filter on waves in the frequency range (0.1, 1.2) for some F with $\Delta t = 1$. However Table 2 shows the maximum phase changes introduced by the

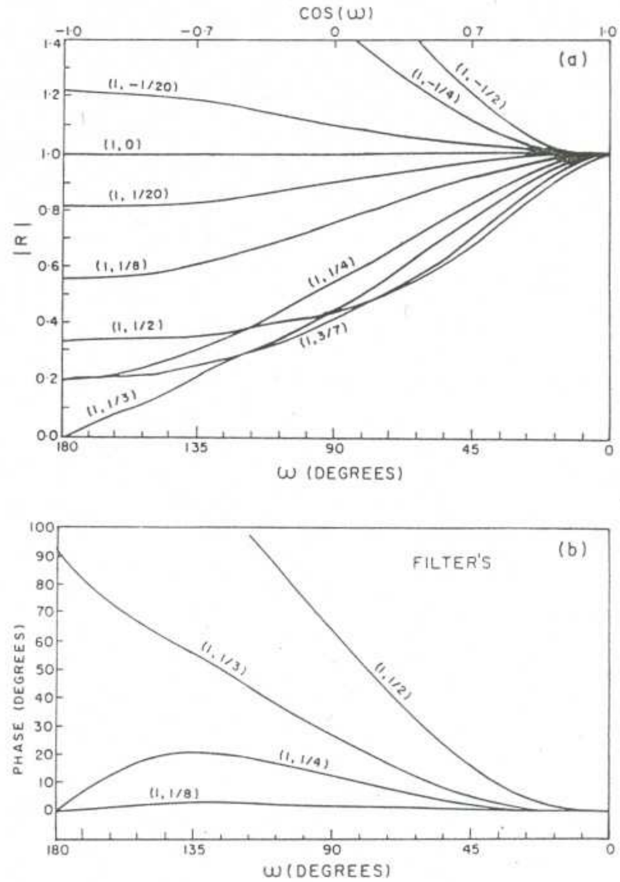

Fig. 1. a) Profiles of $|R|$ vs. ω (in degrees) for certain values of the combination $(\Delta t, F)$. b) Profiles of phase changes (in degrees) introduced by the application of filter alone for certain values of $(\Delta t, F)$

Table 1. *Numerical Values of $|R|$ for Different Combinations of F (⅛, ¼, ⅓, ½) and Δt (1, ¾, ½, ¼) over a Frequency Range of (0.1, 1.2)*

F	Δt	Maximum amplitude	Δt	F	Maximum amplitude
⅛	1	0.9985	1	⅛	0.9985
	¾	0.9991		¼	0.9966
	½	0.9996		⅓	0.9754
	¼	0.9999		½	0.9901
¼	1	0.9966	¾	⅛	0.9991
	¾	0.9981		¼	0.9981
	½	0.9991		⅓	0.9972
	¼	0.9997		½	0.9944
⅓	1	0.9754	½	⅛	0.9996
	¾	0.9972		¼	0.9991
	½	0.9987		⅓	0.9982
	¼	0.9996		½	0.9975
½	1	0.9901	¼	⅛	0.9999
	¾	0.9944		¼	0.9997
	½	0.9975		⅓	0.9996
	¼	0.9993		½	0.9993

Table 2. *Numerical Values of the Phase Changes (in degrees) Introduced by the Frequency Filter for Different Combinations of F (1/8, 1/4, 1/3, 1/2) and Δt (1, 3/4, 1/2, 1/4) over a Frequency Range of (0.1, 1.2)*

F	Δt	Maximum phase change (degrees)	Δt	F	Maximum phase change (degrees)
1/8	1	1.37	1	1/8	1.37
	3/4	0.68		1/4	7.16
	1/2	0.23		1/3	14.88
	1/4	0.03		1/2	39.11
1/4	1	7.16	3/4	1/8	0.68
	3/4	3.58		1/4	3.58
	1/2	1.21		1/3	1.38
	1/4	0.17		1/2	21.95
1/3	1	14.88	1/2	1/8	0.23
	3/4	1.38		1/4	1.21
	1/2	2.62		1/3	2.62
	1/4	0.37		1/2	8.70
1/2	1	39.11	1/4	1/8	0.03
	3/4	21.95		1/4	0.17
	1/2	8.70		1/3	0.37
	1/4	1.39		1/2	1.39

application of this filter on waves in the frequency range (0.1, 1.2) for the aforementioned 16 combinations of F and Δt. It implies from Table 2 that the maximum phase change introduced by this filter 1) decreases as Δt decreases for some fixed F and 2) decreases as F decreases for some fixed Δt. On combining these implications of the Table 1 and Table 2 it could be said that $|R|$ increases while θ decreases for $R = |R| e^{i\theta}$ as 1) Δt decreases with fixed F and 2) F decreases with fixed Δt.

Fig. 2 a shows profiles of the size of amplification factor, $|X|$, for the Adams-Bashforth finite difference scheme against frequency for physical and computational modes for four different combinations of (Δt, F). The figure shows that this scheme amplifies the physical mode and damps the computational mode. Phase changes effected by the application of the Adams-Bashforth scheme in the physical and computational modes for waves in the frequency range (0.1, 1.2) are shown in degrees in Fig. 2 b. Fig. 2 b shows that the frequency range (0.1, 1.2) can be considered as a combination of two subranges such that the phase changes introduced in computational mode are more than that in the physical mode in the lower frequency subrange while phase changes are opposite in the higher frequency subrange. Figs.

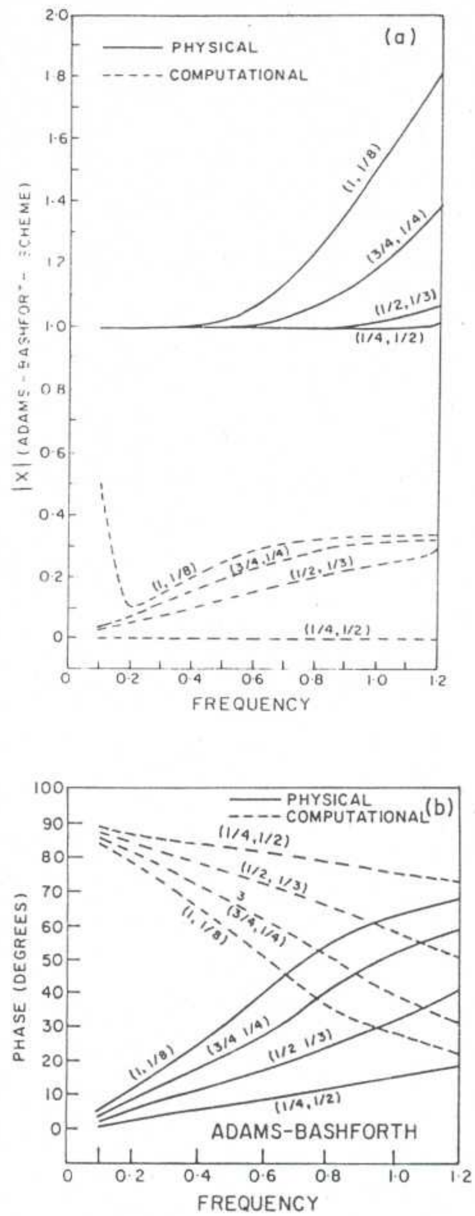

Fig. 2. a) Profiles of amplification factor, $|X|$, for the Adams-Bashforth finite difference scheme versus frequency for some values of (Δt, F). Solid (dashed) lines correspond to the physical (computational) mode. b) Profiles of the phase changes (in degrees) introduced by the Adams-Bashforth scheme in the physical and computational modes versus frequency for some values of (Δt, F)

3 a and 3 b show the size of the amplification factor, $|X|$ for, and the phase changes introduced by, the leap frog finite difference scheme against frequency for physical and computational modes for four different values of the combination (Δt, F), respectively.

Simple advective eq. (15) was integrated for 10 time steps using the Adams-Bashforth finite dif-

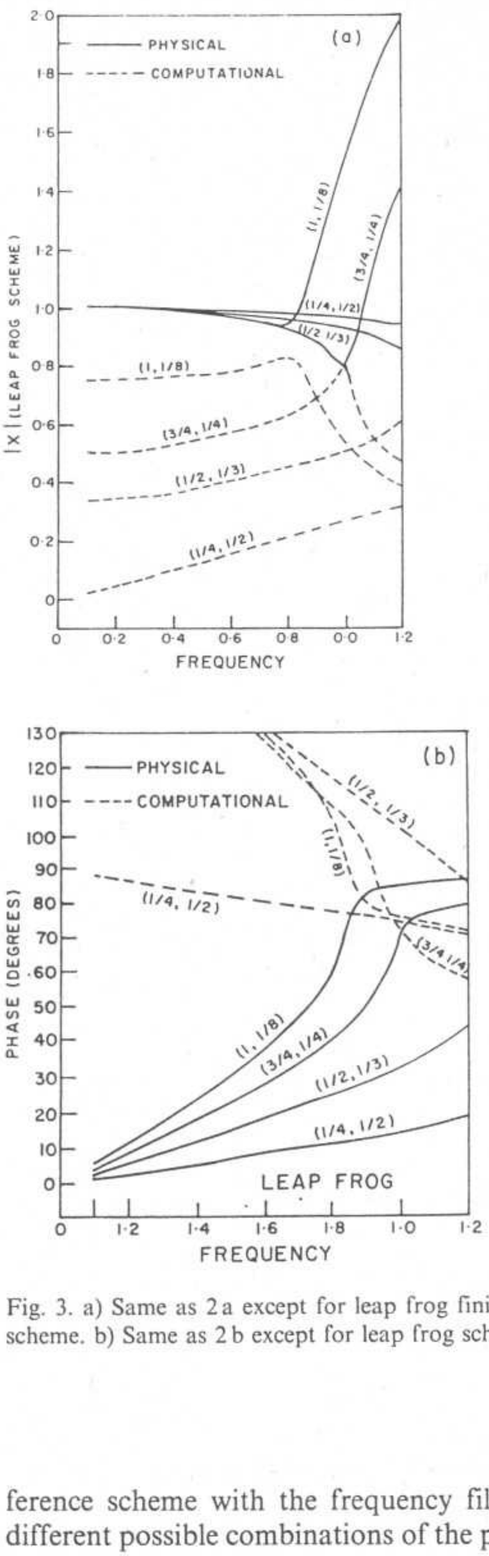

Fig. 3. a) Same as 2a except for leap frog finite difference scheme. b) Same as 2b except for leap frog scheme

Table 3. *Numerical Values of the Maximum Amplitude, $|X|$, for the Adams-Bashforth Scheme with Filter, Used over 10 Time Steps for Different Combinations of F (1/8, 1/4, 1/3, 1/2) and Δt (1, 3/4, 1/2, 1/4) over a Frequency Range of (0.1, 1.2)*

F	Δt	Maximum amplitude	Δt	F	Maximum amplitude
1/8	1	350.24	1	1/8	350.24
	3/4	29.35		1/4	316.37
	1/2	2.40		1/3	292.05
	1/4	1.03		1/2	239.91
1/4	1	316.37	3/4	1/8	29.35
	3/4	31.99		1/4	31.99
	1/2	3.00		1/3	33.09
	1/4	1.05		1/2	33.45
1/3	1	292.05	1/2	1/8	2.40
	3/4	33.09		1/4	3.00
	1/2	3.45		1/3	3.45
	1/4	1.07		1/2	4.37
1/2	1	239.91	1/4	1/8	1.03
	3/4	33.45		1/4	1.05
	1/2	4.37		1/3	1.07
	1/4	1.14		1/2	1.14

ference scheme with the frequency filter for 16 different possible combinations of the parameters ($\Delta t, F$) from F = 1/8, 1/4, 1/3, 1/2 and Δt = 1, 3/4, 1/2, 1/4 and for waves in the frequency range (0.1, 1.2). Maximum amplitude and the maximum phase changes introduced by the Adams-Bashforth scheme filter combination, after 10 steps, are shown in Tables 3 and 4, respectively. From Table 3 it is found the maximum amplitude or rather cumulative amplification decreases as Δt decreases for constant F. For the case when F varies with constant Δt we find the cumulative amplification increases as F decreases for $\Delta t = 1$ and it decreases as F decreases for other fixed Δt. It is not possible to draw similar conclusions regarding the variations of the cumulative maximum phase changes w.r.t. F and Δt from the results obtained and given in Table 4 except for the case when F varies with constant Δt = 1/4. In this particular case it is found that cumulative maximum phase changes decreases as F decreases. It could be concluded from the variations of cumulative amplification w.r.t. F and Δt from Table 3 that F and Δt should have least possible values for controlling the wave's cumulative amplification during the integration procedure. From Table 3 we find that maximum amplitude found after 10 steps of integration is least viz. 1.03 when F and Δt have lowest values, 1/8 and 1/4, respectively.

4. Stability Considerations

In this section we shall see stability aspects of the amplification factors, to be obtained, when semi-implicit and leap-frog finite difference schemes are used with the frequency filter in the time in-

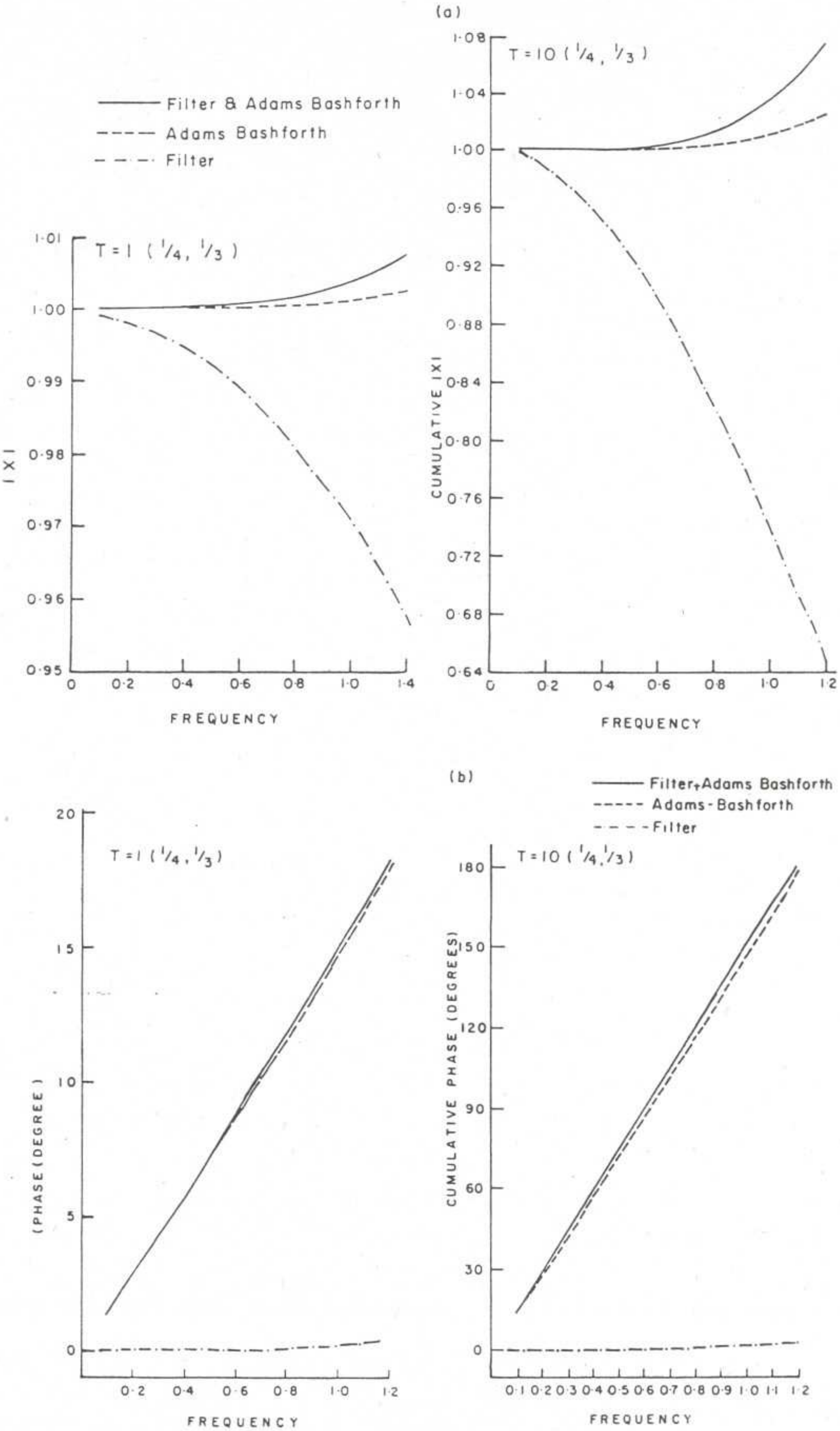

Fig. 4. a) Profiles showing the cumulative effect of the application of the 1. Adams-Bashforth scheme with filter 2. Adams-Bashforth scheme alone and 3. filter alone; on a range of frequencies over 10 time steps. b) Profiles showing cumulative phase changes introduced by the application of 1. Adams-Bashforth scheme with filter 2. Adams-Bashforth scheme alone and 3. filter alone on a range of frequencies over 10 time steps

Table 4. *Numerical Values of the Maximum Phase Changes (in degrees) Introduced by the Successive Use of the Adams-Bashforth Scheme with Filter Over 10 Time Steps for Different Combinations of F (⅛, ¼, ⅓, ½) and Δt (1, ¾, ½, ¼) over a Frequency Range of (0.1, 1.2)*

F	Δt	Maximum phase change (degrees)	Δt	F	Maximum phase change (degrees)
⅛	1	322.19	1	⅛	322.19
	¾	341.23		¼	323.17
	½	322.19		⅓	322.36
	¼	179.77		½	316.13
¼	1	323.17	¾	⅛	341.23
	¾	341.21		¼	341.21
	½	358.91		⅓	339.46
	¼	181.08		½	331.17
⅓	1	322.36	½	⅛	322.19
	¾	339.46		¼	358.91
	½	356.11		⅓	356.11
	¼	182.08		½	345.72
½	1	316.13	¼	⅛	179.77
	¾	331.17		¼	181.08
	½	345.72		⅓	182.08
	¼	184.08		½	184.08

tegration of the simple advective equation (15). Following Kurihara [1965] eq. (15) can be expressed in finite difference form as

$$\frac{\varphi_{t+\Delta t}-\varphi^*_{t-\Delta t}}{2\Delta t} = \mathrm{i}\omega_a\varphi_t + \mathrm{i}(\omega-\omega_a)\left(\frac{\varphi_{t+\Delta t}+\varphi^*_{t-\Delta t}}{2}\right) \tag{27}$$

Now following the procedure adopted in the case of the Adams-Bashforth scheme we get the amplification factor for this case as

$$X = \frac{F+\mathrm{i}\omega_a\Delta t}{1-\mathrm{i}(\omega-\omega_a)\Delta t} \pm \frac{\{(F-1)(F-1+2\omega\omega_a\Delta t^2)+(1-2F)\omega^2\Delta t^2\}^{\frac{1}{2}}}{1-\mathrm{i}(\omega-\omega_a)\Delta t} \tag{28}$$

Here positive sign associated with the radicand corresponds to the physical mode while negative sign corresponds to the computational mode.

Case $\omega_a = 0$ corresponds to the semiimplicit scheme. Thus substitution of $\omega_a = 0$ in eq. (28) leads to the amplification factor for the semi-implificit scheme filter combination as

$$X = \frac{1+\mathrm{i}\omega\Delta t}{1+\omega^2\Delta t^2}[F \pm \{1+\omega^2\Delta t^2+F^2-2F(1+\omega^2\Delta t^2)\}^{\frac{1}{2}}] \tag{29}$$

For stability considerations of this amplification factor we shall confine ourselves to the physical mode only (positive sign). This X will have two different forms depending on the sign of the value of the expression $1+\omega^2\Delta t^2+F^2-2F(1+\omega^2\Delta t^2)$. When it is positive we get: $\omega^4+\omega^2 \geqslant 0$ from $|X| \leqslant 1$, which is always true. The semi-implicit scheme filter combination is stable whenever $1+\omega^2\Delta t^2+F^2 \geqslant 2F(1+\omega^2\Delta t^2)$. For the case when $1+\omega^2\Delta t^2+F^2 < 2F(1+\omega^2\Delta t^2)$ we get

$$X = \frac{1+\mathrm{i}\omega\Delta t}{1+\omega^2\Delta t^2}[F+\{\mathrm{i}2F(1+\omega^2\Delta t^2)-1-\omega^2\Delta t^2-F^2\}^{\frac{1}{2}}] \tag{29a}$$

In this case $|X| \leqslant 1$ leads to an inequality $F \leqslant 1$. For this case, we find that semiimplicit scheme with filter is stable whenever $F \leqslant 1$. As a corollary of this we find that semiimplicit scheme is always stable as substitution of $F=0$, a value well in conformity with $F \leqslant 1$, implies exclusion of filter from the scheme filter combination.

Case $\omega = \omega_a$ corresponds to the leap-frog scheme. Substitution of $\omega = \omega_a$ in eq. (28) gives the amplification factor for the leap-frog scheme filter combination as

$$X = F+\mathrm{i}\omega\Delta t \pm \{(F-1)(F-1+2\omega^2\Delta t^2)+(1-2F)\omega^2\Delta t^2\}^{\frac{1}{2}} \tag{30}$$

For stability aspects we again consider X for two different situations viz. when the expression under the radicand sign is either positive or negative. When it is positive we find that $|X| \leqslant 1$ leads to the inequality $\omega^2\Delta t^2 \geqslant 0$, which is always true. Thus for this particular case we find that the leap-frog scheme filter combination is always stable. For the case when this expression is negative we find that $|X| \leqslant 1$ leads to an inequality

$$\omega\Delta t \leqslant \left(\frac{1-F}{1+F}\right)^{\frac{1}{2}} \tag{31}$$

This is the inequality which must be satisfied by F and Δt for any ω to make the leap-frog scheme filter combination stable for this case. This equation implies that the time step for the leap-frog

Table 5. *Numerical Values of the Maximum Allowable Time Step for Admitting same Range of Frequencies in the Leap-Frog Scheme Filter Combination as in Leap-Frog Scheme Alone of Time Step Δt for Different Values of F and Δt*

F	Δt	Maximum allowable time step
1/8	1	0.882
	3/4	0.661
	1/2	0.441
	1/4	0.221
1/4	1	0.775
	3/4	0.581
	1/2	0.387
	1/4	0.194
1/3	1	0.707
	3/4	0.530
	1/2	0.354
	1/4	0.177
1/2	1	0.577
	3/4	0.433
	1/2	0.288
	1/4	0.144

scheme filter combination, with filter element F, must be reduced to $\left(\frac{1-F}{1+F}\right)^{\frac{1}{2}}$ times Δt if we wish to admit the same range of frequencies in the scheme-filter combination as in the leap-frog scheme alone, having temporal grid Δt, in the stable domain of X. Time step $\Delta t\left(\frac{1-F}{1+F}\right)^{\frac{1}{2}}$ thus obtained is the maximum allowable time step for making the leap-frog scheme filter combination stable. Table 5 gives values for the maximum allowable time steps for 16 different combinations of F and Δt.

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