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ENERGETIC CONSISTENCY OF TRUNCATED MODELS

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1. After the classical work of Lorenz (1960), it is well-established in the field of meteorology that certain terms or groups of terms in the vorticity and divergence equations should either be retained or dropped together to achieve "energetic consistency". This has led to an hierarchy of models beginning with complete vorticity and divergence equation model and ending with quasi-geostrophic model. By integration over closed horizontal surface, we show that the energetic consistency of the various truncated models is of more restricted type than would appear from the form which it takes for the atmospheric mass as a whole. In general, these truncated models create fictitious cancellations of vertical divergences of energy fluxes at each level and hence vertical coupling of energy in these truncated models is defective.

2. Equation of quasi-static frictionless horizontal motion is

$$\frac{d\vec{V}}{dt} + f\,\vec{k} \times \vec{V} + \nabla\phi = 0 \qquad \ldots \qquad (1)$$

$\vec{V}$ is horizontal velocity vector; ∇ is horizontal operator and $\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \vec{V}.\nabla + \omega\frac{\partial}{\partial p}$ is total individual derivative.

By performing $\nabla.$ and $\vec{k}.\nabla\times$ operations on this equation, we get the well-known divergence and vorticity equations :

$$\frac{\partial D_3}{\partial t} + V_2 . \nabla D_3 + \vec{V}_3 . \nabla D_3 + \omega_3 \frac{\partial D_3}{\partial p} + D_3^2 + 2J(v_2, u_2) +$$
$$+ 2J(v_3, u_3) + 2J(v_2, u_3) + 2J(v_3, u_2) - f\zeta_2 + u_2\beta +$$
$$+ u_3\beta + \nabla\omega_3 . \frac{\partial \vec{V}_3}{\partial p} + \nabla\omega_3 . \frac{\partial \vec{V}_2}{\partial p} + \nabla^2\phi_1 = 0 \qquad \ldots \qquad (2)$$

$$\frac{\partial \zeta_2}{\partial t} + \vec{V}_2 . \nabla \zeta_2 + \vec{V}_3 . \nabla \zeta_2 + \omega_3 \frac{\partial \zeta_2}{\partial p} + v_2 \beta + v_3 \beta + f D_3 +$$

$$+ \zeta_2 D_3 + J \left(\omega_3 , \frac{\partial \chi_3}{\partial p} \right) + \nabla \omega_3 . \nabla \frac{\partial \psi_2}{\partial p} = 0 \quad \ldots \quad (3)$$

Following Lorenz (1960), we have used subscript $_1$ to indicate a physical property and subscripts $_2$ and $_3$ to indicate quantities derived from stream-function ψ_2 and velocity potential χ_3 respectively.

We multiply each term of the vorticity equation by $-\psi_2$ and each term of divergence equation by $-\chi_3$ and integrate over the earth's closed horizontal spherical surface. We do not perform vertical integration and in this respect deviate from the analysis of Lorenz (1960). We freely use the property that divergence of a horizontal vector and horizontal Jacobian of two scalars vanish when integrated over closed horizontal surface. The terms which result after some simple manipulations are shown in Table 1. We use the notation

$$k_2 \equiv \frac{\vec{V}_2 . \vec{V}_2}{2} \; ; \quad k_3 \equiv \frac{\vec{V}_3 . \vec{V}_3}{2} \; ; \quad \vec{V} = \vec{V}_2 + \vec{V}_3$$

3. We now have from vorticity and divergence equations

$$\int_S \left\{ \frac{\partial k_2}{\partial t} + \eta \nabla \psi_2 . \nabla \chi_3 + \omega_3 \left(\frac{\partial k_2}{\partial p} + \vec{V}_2 . \frac{\partial \vec{V}_3}{\partial p} \right) \right\} dS = 0$$

$$\int_S \left\{ \frac{\partial k_3}{\partial t} - \eta \nabla \psi_2 . \nabla \chi_3 + \omega_3 \left(\frac{\partial k_3}{\partial p} + \vec{V}_3 . \frac{\partial \vec{V}_2}{\partial p} \right) + \vec{V} . \nabla \left(\vec{V}_2 . \vec{V}_3 + k_2 + k_3 + \phi_1 \right) \right\} dS = 0$$

From combination of these two energy equations, we get

$$\int_S \left\{ \frac{\partial}{\partial t} \left(k_2 + k_3 + \vec{V}_2 . \vec{V}_3 \right) - \frac{\partial}{\partial t} \left(\vec{V}_2 . \vec{V}_3 \right) + \vec{V} . \nabla \phi_1 \right\} dS = 0$$

Now $\displaystyle\int_S \vec{V}_2 . \vec{V}_3 \, dS = \int_S J (\psi_2 , \chi_3) \, dS = 0$

$$\therefore \frac{\partial}{\partial t} \int_S \vec{V}_2 . \vec{V}_3 \, dS = 0$$

$$\therefore \int_S \left\{ \frac{d}{dt} \left(k_2 + k_3 + \vec{V}_2 \cdot \vec{V}_3 \right) + \vec{V} \cdot \nabla\phi_1 \right\} dS = 0$$

$$\text{i.e.} \int_S \left(\frac{d}{dt} \frac{\vec{V}^2}{2} + \vec{V} \cdot \nabla\phi_1 \right) dS = 0 \qquad \ldots \qquad (4)$$

This is the dynamically consistent energy equation which we should expect straight from the equation of motion (1) after its dot-multiplication by $\vec{V}$. It can be shown that if we further integrate equation (4) with respect to p in the vertical, we shall get for adiabatic frictionless flow :

$$\frac{\partial}{\partial t} \left(P_1 + I_1 + K_2 + K_3 \right) = 0 \qquad \ldots \qquad (5)$$

where $P_1 = \int_M gz \; dM$

$I_1 = \int_M C_v \, T \, dM$

$K_2 = \int_M \frac{\vec{V}_2^2}{2} \, dM$

$K_3 = \int_M \frac{\vec{V}_3^2}{2} \, dM$

$K = \int_M \frac{\vec{V}^2}{2} \, dM$

If energy equation (4) is satisfied in each plane, then it can be shown that energy equation (5) is necessarily satisfied over the atmospheric mass as a whole. But the converse is not true. What the truncated models do is to satisfy (5) with omission of K_3, while they satisfy (4) in more restricted forms. For example, (3,3) terms in divergence equation yield $\frac{\partial}{\partial p}(\omega_3 k_3)$; (2,3) terms of divergence equation and (3,3) terms of vorticity equation cumulatively yield $\frac{\partial}{\partial p}\left\{ \omega_3 (\vec{V}_2 . \vec{V}_3) \right\}$; (2,2) terms in divergence equation and (2,3) terms in vorticity equation together yield $\frac{\partial}{\partial p}(\omega_3 k_2)$. Now the vertical divergence of these vertical energy flux terms need not vanish at each horizontal level, although on vertical integration w.r.t. pressure, these terms make zero contribution on assumption of $\omega_3 = 0$ at top and bottom of the atmosphere. As such, these truncated models create fictitious cancellations of vertical divergence of energy fluxes at each level and hence vertical coupling of energy in these truncated models is defective. The result of integration over entire mass of the atmosphere is also shown in Table 1.

4. We shall now examine the energetics of the various truncated models in respect of the four energy equations :

i) from vorticity equation in a plane,

ii) from divergence equation in a plane,

iii) from their combination in a plane and

iv) from their combination over the entire mass of the atmosphere.

4.1 Truncated Model I : Omit $\frac{\partial D_3}{\partial t}$ from divergence equation, retaining all the terms of the vorticity equation.

Then, we have

$$\text{i)} \int_S \left\{ \frac{\partial k_2}{\partial t} + \eta \, \nabla\psi_2 . \nabla\chi_3 + \omega_3 \left(\frac{\partial k_2}{\partial p} + \vec{V}_2 . \frac{\partial \vec{V}_3}{\partial p} \right) \right\} dS = 0$$

$$\text{ii)} \int_S \left\{ -\eta \nabla\psi_2 \cdot \nabla\chi_3 + \omega_3 \left(\frac{\partial k_3}{\partial p} + \vec{V}_3 \cdot \frac{\partial \vec{V}_2}{\partial p} \right) + \vec{V} \cdot \nabla \left(\vec{V}_2 \cdot \vec{V}_3 + k_2 + k_3 + \phi_1 \right) \right\} dS = 0$$

$$\text{iii)} \int_S \left\{ \left(\frac{d}{dt} \frac{\vec{V}^2}{2} + \vec{V} \cdot \nabla \phi_1 \right) - \frac{\partial k_3}{\partial t} \right\} dS = 0$$

$$\text{iv)} \frac{\partial}{\partial t} (P_1 + I_1 + K_2) = 0$$

4.2 Truncated Model II : Further, omit $u_3\beta$ and (3,3) terms from divergence equation, still retaining all the terms of the vorticity equation. Then

$$\text{i)} \int_S \left\{ \frac{\partial k_2}{\partial t} + \eta \nabla\psi_2 \cdot \nabla\chi_3 + \omega_3 \left(\frac{\partial k_2}{\partial p} + \vec{V}_2 \cdot \frac{\partial \vec{V}_3}{\partial p} \right) \right\} dS = 0$$

$$\text{ii)} \int_S \left\{ -\eta \nabla\psi_2 \cdot \nabla\chi_3 + \omega_3 \left(\vec{V}_3 \cdot \frac{\partial \vec{V}_2}{\partial p} \right) + \vec{V} \cdot \nabla \left(\vec{V}_2 \cdot \vec{V}_3 + k_2 + \phi_1 \right) \right\} dS = 0$$

$$\text{iii)} \int_S \left[\left(\frac{d}{dt} \frac{\vec{V}^2}{2} + \vec{V} \cdot \nabla\phi_1 \right) - \left\{ \frac{\partial k_3}{\partial t} + \frac{\partial}{\partial p} (\omega_3 k_3) \right\} \right] dS = 0$$

$$\text{iv)} \frac{\partial}{\partial t} (P_1 + I_1 + K_2) = 0$$

4.3 Truncated Model III : Further, omit (2,3) terms from divergence equation and $J\left(\omega_3, \frac{\partial \chi_3}{\partial p}\right)$ terms from vorticity equation. Then the divergence equation becomes the balance equation. Now the divergence and vorticity equations are :

$$2J(v_2, u_2) + (u_2\beta - f\zeta_2) + \nabla^2\phi_1 = 0$$

$$\frac{\partial \zeta_2}{\partial t} + (\vec{V}_2 \cdot \nabla\zeta_2 + v_2\beta) + \left(\vec{V}_3 \cdot \nabla\zeta_2 + \zeta_2 D_3 + \omega_3 \frac{\partial \zeta_2}{\partial p} + \nabla\omega_3 \cdot \nabla \frac{\partial \psi_2}{\partial p} \right) + (v_3\beta + fD_3) = 0$$

The four energy equations become :

$$\text{i)} \int_S \left(\frac{\partial k_2}{\partial t} + \eta \nabla\psi_2 \cdot \nabla\chi_3 + \omega_3 \frac{\partial k_2}{\partial p} \right) dS = 0$$

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ii) $$\int_S \left(-\eta\ \nabla\psi_2 . \nabla\chi_3 + \vec{V} . \nabla k_2 + \vec{V} . \nabla\phi_1 \right) dS = 0$$

iii) $$\int_S \left[\left(\frac{d}{dt} \frac{\vec{V}^2}{2} + \vec{V} . \nabla\phi_1 \right) - \left\{ \frac{\partial k_3}{\partial t} + \frac{\partial}{\partial p} \left(\omega_3 k_3 + \omega_3 \vec{V}_2 \vec{V}_3 \right) \right\} \right] dS = 0$$

iv) $$\frac{\partial}{\partial t} (P_1 + I_1 + k_2) = 0$$

4.4 Truncated model IV : Further, omit $2J(v_2, u_2)$ term from the divergence equation and (2, 3) terms from vorticity equation. Then the divergence and vorticity equations become

$$u_2\beta - f\zeta_2 + \nabla^2\phi_1 = 0$$

$$\frac{\partial\zeta_2}{\partial t} + \vec{V}_2 . \nabla\zeta_2 + v_2\beta + v_3\beta + f\ D_3 = 0$$

The four energy equations become :

i) $$\int_S \left(\frac{\partial k_2}{\partial t} + f\ \nabla\ \psi_2 . \nabla\chi_3 \right) dS = 0$$

ii) $$\int_S \left(-f\ \nabla\psi_2 . \nabla\chi_3 + \vec{V} . \nabla\phi_1 \right) dS = 0$$

iii) $$\int_S \left[\left(\frac{d}{dt} \frac{\vec{V}^2}{2} + \vec{V} . \nabla\phi_1 \right) - \left\{ \frac{\partial k_3}{\partial t} + \frac{\partial}{\partial p} \left(\omega_3 k_3 + \omega_3 \vec{V}_2 . \vec{V}_3 + \omega_3 k_2 \right) \right\} \right] dS = 0$$

iv) $$\frac{\partial}{\partial t} (P_1 + I_1 + k_2) = 0$$

4.5 Truncated Model V : Further, neglect variation of f in dealing with expression $(u_2\beta - f\zeta_2)$ in divergence equation and $(v_3\beta + fD_3)$ in vorticity equation. Now the divergence and vorticity equations are :

$$-f_0\zeta_2 + \nabla^2\phi_1 = 0$$

$$\frac{\partial\zeta_2}{\partial t} + \vec{V}_2 . \nabla\zeta_2 + v_2\ \beta + f_0\ D_3 = 0$$

The four energy equations are the same as for Truncated Model-IV above.

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$-\oint_s \chi_3$(Divergence eq.) ds $\qquad -\oint_s \psi_2$(Vorticity eq.) ds

Divergence Equation		Vorticity Equation		Serial Number of group	Total contribution of group after integration over	
Term of Divergence Eq.	contribution to $\frac{\partial k}{\partial t}$	Term of vorticity Eq.	contribution to $\frac{\partial k}{\partial t}$		closed horizontal surface	entire mass of atmosphere
$\frac{\partial D_3}{\partial t}$	$\frac{\partial k_3}{\partial t}$	$\frac{\partial \xi_2}{\partial t}$	$\frac{\partial k_2}{\partial t}$	1	$\oint_s \frac{\partial}{\partial t}(k_2+k_3)\, ds$	$\frac{\partial}{\partial t}(K_2+K_3)$
		$\mathbf{V}_2 \cdot \nabla \xi_2$ $v_2 \beta$	0 0	2	0	0
$u_3 \beta$	0			3	0	0
$\mathbf{V}_3 \cdot \nabla D_3$ D_3^2 $2J(v_3, u_3)$ $\omega_3 \frac{\partial D_3}{\partial p}$ $\nabla \omega_3 \cdot \frac{\partial \mathbf{V}_3}{\partial p}$	$\chi_3 D_3^2 - 2\mathbf{V} \cdot \nabla k_3$ $-\chi_3 D_3^2$ $3\mathbf{V} \cdot \nabla k_3$ $-\chi_3 \omega_3 \frac{\partial D_3}{\partial p}$ $\chi_3 \omega_3 \frac{\partial D_3}{\partial p} + \omega_3 \frac{\partial k_3}{\partial p}$			4	$\oint_s (\mathbf{V} \cdot \nabla k_3 + \omega_3 \frac{\partial k_3}{\partial p})\, ds$ Or $\oint_s \frac{\partial}{\partial p}(\omega_3 k_3)\, ds$	0
$\mathbf{V}_2 \cdot \nabla D_3$ $2J(v_2,u_3)+2J(v_3,u_2)$ $\nabla \omega_3 \cdot \frac{\partial \mathbf{V}_2}{\partial p}$	$-\mathbf{V} \cdot \nabla(\mathbf{V}_2 \cdot \mathbf{V}_3)$ $2\mathbf{V} \cdot \nabla(\mathbf{V}_2 \cdot \mathbf{V}_3)$ $\omega_3 \mathbf{V}_3 \cdot \frac{\partial \mathbf{V}_2}{\partial p}$	$J(\omega_3, \frac{\partial \chi_3}{\partial p})$	$\omega_3 \mathbf{V}_2 \cdot \frac{\partial \mathbf{V}_3}{\partial p}$	5	$\oint_s \{\mathbf{V} \cdot \nabla(\mathbf{V}_2 \cdot \mathbf{V}_3) + \omega_3 \frac{\partial}{\partial p}(\mathbf{V}_2 \cdot \mathbf{V}_3)\}\, ds$ Or $\oint_s \frac{\partial}{\partial p}\{\omega_3(\mathbf{V}_2 \cdot \mathbf{V}_3)\}\, ds$	0
$2J(v_2, u_2)$	$\mathbf{V} \cdot \nabla k_2 - \xi \nabla \psi_2 \cdot \nabla \chi_3$	$\mathbf{V}_3 \cdot \nabla \xi_2$ $\xi_2 D_3$ $\omega_3 \frac{\partial \xi_2}{\partial p}$ $\nabla \omega_3 \cdot \nabla \frac{\partial \psi_2}{\partial p}$	$\psi_2 \xi_2 D_3 + \xi_2 \nabla \psi_2 \cdot \nabla \chi_3$ $-\psi_2 \xi_2 D_3$ $-\psi_2 \omega_3 \frac{\partial \xi_2}{\partial p}$ $\psi_2 \omega_3 \frac{\partial \xi_2}{\partial p} + \omega_3 \frac{\partial k_2}{\partial p}$	6	$\oint_s \mathbf{V} \cdot \nabla k_2 + \omega_3 \frac{\partial k_2}{\partial p}\, ds$ Or $\oint_s \frac{\partial}{\partial p}(\omega_3 k_2)\, ds$	0
$u_2 \beta$ $-f \xi_2$	$-\chi_3 u_2 \beta$ $\chi_3 u_2 \beta - f \nabla \psi_2 \cdot \nabla \chi_3$	$v_3 \beta$ $f D_3$	$-\psi_2 v_3 \beta$ $\psi_2 v_3 \beta + f \nabla \psi_2 \cdot \nabla \chi_3$	7	0	0
$\nabla^2 \phi_1$	$\mathbf{V} \cdot \nabla \phi_1$			8	$\oint_s \mathbf{V} \cdot \nabla \phi_1\, ds$	$\frac{\partial}{\partial t}(P_1 + I_1)$
$\nabla \cdot \mathbf{F}$	$\mathbf{V}_3 \cdot \mathbf{F}$	$\mathbf{k} \cdot \nabla \times \mathbf{F}$	$\mathbf{V}_2 \cdot \mathbf{F}$	9	$\oint_s \mathbf{V} \cdot \mathbf{F}\, ds$	$\oint_M \mathbf{V} \cdot \mathbf{F}\, dM$