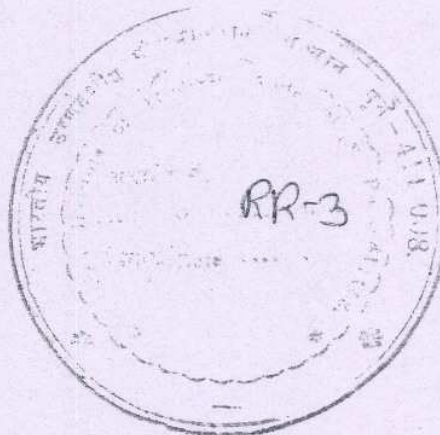# SIMULATION OF THE SPECTRAL CHARACTERISTICS OF THE LOWER ATMOSPHERE BY A SIMPLE ELECTRICAL MODEL AND USING IT FOR PREDICTION

by

Subroto Sinha

INDIAN INSTITUTE OF TROPICAL METEOROLOGY

Ramdurg House
Ganeshkhind Road,
Poona-5, India

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List of Symbols

a = acceleration of the rigid support

f = viscous force acting on a mass m

K = stiffness factor of the spring

$H(S)$ = Laplace transform of the system transfer function

x = space cordinate $\quad t$ = time cordinate

$X(S)$ = Laplace transform of the system response

$U(t)$ = unit impulse function

q = instantaneous electric charge

I = instantaneous electric current

$u_i(\vec{x}, t)$ = velocity vector where $i = 1, 2, 3$

$\vec{k}$ = wave number vector

$\vec{Z}(\vec{k}, t)$ = another random vector

$dZ_i(\vec{k}, t)$ = differential of $\vec{Z}(\vec{k}, t)$, $i = 1, 2, 3$

$dZ_i^*(\vec{k}, t)$ = complex conjugate of $dZ_i(\vec{k}, t)$

$\phi_{ij}(\vec{k})$ = spectral tensor density of a homogeneous isotropic velocity field.

$\psi_{ij}(\vec{k})$ = transverse component of $\phi_{ij}(\vec{k})$

$\eta_{ij}(\vec{k})$ = longitudinal component of $\phi_{ij}(\vec{k})$

ϵ = mean rate of dissipation of energy

ν = coefficient of viscosity of the medium

$\beta = \dfrac{f}{2(km)^{\frac{1}{2}}}$ = dimensionless variable representing the damping ratio

$\omega_n = \left(\dfrac{K}{m}\right)^{\frac{1}{2}}$ = undampedn natural frequency of the system

ω_m = frequency at which the quantity $\dfrac{\omega \phi_0(\vec{k})}{\sigma_u^2}$ maximum

σ_u = standard deviation of the longitudinal component of velocity

$u_0(t)$ = output function from the L, C, R circuit

$\phi_0(\omega)$ = spectral density of output function

$\eta_0(\omega)$ = spectral density of output function when the longitudinal component of velocity function is fed at input

η_1 = constant upper limit of $\eta_0(\omega)$

t_1 = time at which the input random function starts

η_{in} = spectral density of input longitudinal velocity component

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L E G E N D

Fig. 1   :   A simple version of an accelerometer

a - rigid support,

b - mass m,

c - spring of stiffness factor K

Fig. 2 :   A series L, C, R circuit

Fig. 3 :   The transient response of the L, C, R, circuit to
a unit-impulse. Along the abcissa are plotted the
dimensionless variable $\dfrac{\omega_n t}{2 \Pi}$ and along the ordinate
the amplitude ratio. (After Gill, J.C, Pelegrin, M.J.,
and Decanline, P. 1959 (2)).

Fig. 4 :   Representation of a random function by a series of
impulses.

Simulation of the spectral characteristics of the lower atmosphere by a
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Summary

This paper attempts to bring out the broad features of the spectral
characteristics of the lowest 300 metres of the atmosphere, as observed by
Ivanov and Ordanovich [4]. The various atmospheric parameters are simulated
by the elements of a mechanical system which is then converted into its
electric analog. The magnitude and the frequency of occurrence of the
resonance value of the response of the system are discussed, using the
differential equation representing the system. The components of the wind
velocity, treated as a random function of time, is fed at the input of the
system, and a scheme is put forward for predicting future values of the
random function.

1. Introduction

In [4] the authors have presented the results of their investigations of the wind velocity spectrum for unstable stratification in the lower 300-meter layer of the atmosphere. Their main aim was to study the low frequency range of the spectrum, upto periods of the order of 30 minutes. They therefore used a filter whose transfer function had the form :-

$$H (\omega) \approx \frac{1}{1 + (\omega \tau)^2}$$

where ω is the angular frequency and τ is the time-constant, which in their measurements was equal to approximately 60 secs. The high frequency components were thus filtered out and thereby the specific weight of the low frequency components increased in total energy.

The measurements were carried out on a tall meteorological tower of the Institute of Applied Geophysics in Obninsk. A set of propeller anemometers of the M-49 type were used to measure the longitudinal component. Bivanes, to measure the vertical and horizontal angles which the wind velocity vector forms with the mean direction of air flow, were used to obtain the vertical and transverse components. The longitudinal component sensors were mounted on 12 levels, and the Bivanes at 8 levels encompassing the lower 300-meter layer of the atmosphere including the 4 and 8 m levels for which measurements were carried out on a small meteorological tower.

The normalized spectral densities obtained from these measurements showed the presence of two pronounced maxima. The first of these was greater in magnitude and corresponded to a period of 15-20 mins. The

authors ascribed this maximum as being related to the action of buoyancy forces. The second maximum which occurred only for the longitudinal component, had a period of 5-8 mins. This was related to the action of frictional forces. The observations were divided into four series, categorised by the Monin-Obukhov length, L, which had the values, -150, -1000, -6, -12, respectively. When, 'L' had the value -1000, the first maximum shifted towards higher frequency.

Observation made earlier by Webb [7], had also shown similar features of the energy spectrum, with a major peak at the low frequency and followed by a number of secondary peaks of much smaller magnitudes, at the higher frequencies. This study was made for an altitude of 29 m. under different wind conditions, without the use of any high-frequency filter.

Representation :- The velocity vector $u_i(\vec{x}, t)$ can be represented in the form of a Fourier. Stieltjes' integral as :-

$$u_i(\vec{x}, t) = \int e^{\sqrt{-1}\,\vec{k}.\vec{x}}\, dZ_i(\vec{k}, t) \qquad \ldots \qquad (1.1)$$

The increments $dZ_i(\vec{k}, t)$ at any value of $\vec{k}$ depends on the particular realization of the velocity distribution such that $dZ_i^*(\vec{k}', t)\, dZ_i(\vec{k}'', t)$ is zero unless $\vec{k}'$ and $\vec{k}''$ are equal. The spectral density is given by :

$$\int \phi_{ij}(\vec{k})\, d\vec{k} = \overline{u_i^*(\vec{x}, t)\, u_j(\vec{x}, t)} \qquad \ldots \qquad (1.2)$$

the integration being taken over all $\vec{k}$ space. The spectral densities of the transverse and longitudinal components of velocity in a homogeneous compressible fluid has been shown by Moyal [3] to be given by :-

$$\tfrac{1}{2}\, \phi_{ii}(\vec{k}) = \phi_{ij}(\vec{k}) = \psi_{ij}(\vec{k}) + \eta_{ij}(\vec{k})$$

The mean rate of dissipation of energy by viscosity was shown by Moyal [3] to be :-

$$\epsilon = 2 \nu \int\!\!\!\int \left\{ \psi(\vec{k}) + \frac{4}{3}\eta(\vec{k}) \right\} k^2 d\vec{k}$$

$$= 8 \Pi \nu \int_0^\infty \left\{ \psi(\vec{k}) + \frac{4}{3} \eta(\vec{k}) \right\} k^4 dk \qquad \ldots \quad (1.4)$$

2. Formulation of the Model

The characteristics of the lower atmosphere show that its behaviour is governed by three major parameters:-

(a) The lapse-rate of the unstable atmosphere and its deviation from the adiabatic lapse-rate. This determines the buoyancy forces which will act on an air parcel.

(b) The viscosity of the medium, which determines the frictional force acting on the air parcel.

(c) The acceleration of the air parcel which determines the inertial force on it.

This behaviour may be likened to a mechanical system which is a simplified version of an accelerometer, shown in Fig.(1). It consists of a mass 'm' in horizontal rectilinear motion along the x-direction, with respect to a support, to which it is attached by means of a spring. The spring develops a force proportional to its elongation. The viscosity of the medium develops a force proportional to the velocity of the mass relative to its support. The motion of the mass, relative to its support, is represented by the equation :-

$$m \frac{d^2x}{dt^2} = -Kx - f \frac{dx}{dt} + ma \qquad \ldots \qquad \ldots \qquad (2.1)$$

When the support is excited by an unit impulse, the above equation may be

rewritten as :-

$$\frac{d^2x}{dt^2} + 2\beta\omega_n \frac{dx}{dt} + \omega_n^2 x = \omega_n^2 U(t) \qquad \ldots \qquad (2.2)$$

where the unit-impulse is written in the particular form for reasons of homogeneity.

Assuming the system to be initially at rest, the Laplace transform of its response to a unit-impulse is given by :-

$$X(S) = \frac{\omega_n^2}{S^2 + 2\beta\omega_n S + \omega_n^2}$$

$$= H(S) \qquad \ldots \qquad (2.3)$$

Converting this mechanical system into its electrical analog by putting mass m equivalent to inductance L, mechanical resistance f equivalent to electrical resistance R, and stifness K equivalent to the reciprocal of capacitance C Equation (2.2) can be rewritten as :-

$$\frac{d^2q}{dt^2} + \frac{R}{L} \frac{dq}{dt} + \frac{1}{LC} q = \frac{1}{LC} U(t) \qquad \ldots \qquad (2.4a)$$

or

$$\frac{dI}{dt} + \frac{R}{L} I + \frac{1}{LC} \int I\, dt = \frac{1}{LC} U(t) \qquad \ldots \qquad (2.4b)$$

where the constants β and ω_n now have the values :

$$\beta = \frac{R}{2} \sqrt{\frac{C}{L}} \quad \text{and} \quad \omega_n = \sqrt{\frac{1}{LC}}$$

Equation (2.4a) and (2.4b) represent a simple L, C, R circuit in a series connexion as shown in Fig. (2). The solution of Equation (2.4a) can be discussed under 3 cases;

viz. :-

Case I $\beta < 1$

$$q = 1 + A\,e^{-\beta\omega_n t}\,\operatorname{Sin}\left[\omega_n(1-\beta)^{\frac{1}{2}}\,t + \theta\right] \tag{2.5a}$$

or $I = \dfrac{dq}{dt} = A\,\omega_n\,e^{-\beta\omega_n t}\,\operatorname{Sin}\left[\omega_n(1-\beta)^{\frac{1}{2}}\,t\right]$ (2.5b)

where $\tan\theta = \dfrac{(1-\beta^2)^{\frac{1}{2}}}{\beta}$, $\operatorname{Cos}\theta = -\beta$

and $A = -\dfrac{1}{\operatorname{Sin}\theta} = (1-\beta^2)^{-\frac{1}{2}}$

This gives an oscillatory response and consists of oscillations that have a frequency $\omega_n(1-\beta)^{\frac{1}{2}}$ and amplitude that is damped by the exponential term $e^{-\beta\omega_n t}$ (see curve (1), Fig. 3).

Case II $\beta = 1$

$$q = 1 - e^{-\omega_n t}(1 + \omega_n t) \tag{2.6}$$

The system is now critically damped. (curve (3), Fig. 3).

Case III $\beta > 1$

$$q = 1 + A\,e^{-\alpha t} + B\,e^{-rt} \tag{2.7}$$

where $-\alpha$ and $-r$ are the roots of the equation

$$S^2 + 2\beta\omega_n S + \omega_n^2 = 0$$

and $A = \dfrac{r}{\alpha - r}$ $B = \dfrac{\alpha}{\alpha - r}$ $\alpha, r > 0$

The response is now non-periodic (curve (4), Fig.3).

Case I gives the condition for the transient response of the circuit to a unit impulse, to be oscillatory. In the present model

the random velocity function $u_i(t)$ is divided into elements of infinitermal width dt, as shown in Fig. 4. At a time $t = \mu$, for example, the element has the width $d\mu$ and the height $u_i(\mu)$. If this element was a unit impulse, the output due to this element alone, would be $H(t - \mu)$, where the transfer function H is given by equation (2.3). Because of it non-unity area, $u_i(\mu)\,d\mu$, the output of the system due to this impulse would be :

$$du_o(t) = u_i(\mu)\,d\mu\; H(t - \mu) \qquad\qquad (2.8)$$

The total output can be obtained by summing up all the infinitesimal outputs due to all the elements into which $u_i(t)$ is divided, from the begining to time t.

$$u_o(t) = \int_{-t_1}^{t} u_i(\mu)\; H(t - \mu)\; d\mu$$

putting $1 = t - \mu$ we have

$$u_o(t) = \int_{o}^{t+t_1} H(1)\; u_i(t - 1)\; d1 \qquad\qquad (2.9)$$

The random input velocity function is thus simulated by a succession of positive and negative impulses, of varying amplitudes. The time interval between two consecutive impulses is less than the time required for the transient to die away. From Fig.3, it is seen that for $\beta = 0.1$, the output from the system would be a superposition of sinusoidal waves of varying amplitudes and frequencies.

Resonant Frequency

The resonant frequency of an oscillatory system is the frequency ω_R at which the modulus of the transfer function is maximum. For the system represented by equation (2.4), we have

$$\omega_R = \omega_n \left(1 - 2 \beta^2 \right)^{\frac{1}{2}} \tag{2.10}$$

the transfer function of the system is related to the spectral densities of the inputs and outputs of the system for the longitudinal component by the equation

$$\eta_0 \, (\vec{k}) = \left[\, H \, (\omega) \, \right]^2 \, \eta_{in} \, (\vec{k}) \tag{2.11}$$

3. Working of the Model

The velocity component $u_i(t)$ is simulated by a series of rectangular pulses, of small finite width whose amplitude is modulated in accordance with the sinosoidal noise voltage from the white noise generator. The pulses will therefore have the same width but varying amplitudes and frequencies. The time interval between successive pulses is less than the time required for the transient response of the L, C, R circuit to die out. This pulse-wave is fed to the L, C, R circuit. The spectral densities of the input and the output signals can be obtained by feeding them separately to a correlator coupled to a high-speed differential analyser. The resultant pattern can be obtained on the screen of a C.R.O.

4. Discussion

In the present model, a reduction in the buoyancy term is simulated by reducing the value of K. This causes a decrease in ω_n because 'm' also increases. The value of β is assumed to remain constant, so that from equation (2.10) it is obvious that the resonant frequency ω_R will decrease.

It is seen from the paper by Webb [7] that the height of the first peak of the energy spectrum is, in general, directly proportional to the turbulent kinetic energy of the wind component. An increase in $\omega_m \eta_o (\vec{k})$ is therefore associated with an increase in σ_u^2 so that the ratio $\omega_m \eta_o (\vec{k})/\sigma_u^2$ can be taken to be approximately constant. In the present model, an increase in the turbulent kinetic energy is simulated by an increase in the amplitudes of the random wave fluctuations. The maximum value of the autocorrelation function, $R(\tau)$ is also thus increased. Since the spectral density is a Fourier transform of the autocorrelation function, its maximum value will also increase, so that the ratio $\dfrac{\omega_m \eta_o (\vec{k})}{\sigma_u^2}$ can be assumed constant.

From equation (2.3) it is seen that

$$\frac{\partial H(S)}{\partial \omega_n} = \frac{\omega_n \left[2 S^2 + 2 \beta \omega_n S\right]}{\left[S^2 + 2\beta \omega_n S + \omega^2\right]^2} \tag{4.1}$$

The R.H.S. of this equation is positive for all positive values of S and ω_n. A decrease in ω_n therefore leads to a decrease in

H(S). Equation (2.11) shows that for a particular value of $\eta_{in}(\vec{k})$, the value of $\eta_o(\vec{k})$ will decrease, following a decrease in ω_n. Since the ratio $\omega_m \eta_o(\vec{k})/\sigma_u^2$ is constant, ω_m must increase. The maximum value of the normalized spectral density will now occur at a higher frequency.

The transverse component of the wind velocity can also be simulated as above, with the difference that the amplitudes in this case would be comparatively smaller. The secondary peaks in the response trace would become very much smaller, to be easily distinguishable.

From Equation (2.3) we also get,

$$\frac{\partial H(S)}{\partial \beta} = -\frac{2\,\omega_n^3\,S}{\left[S^2 + 2\beta\,\omega_n\,S + \omega_n^2\right]^2}$$

An increase in the energy dissipation by friction is simulated by an increase in β. From the above equation it is clear that an increase in β would lead to a decrease in H(S), which would result in a decrease in the output spectral density for a fixed input spectral density.

But from equation (1.4) and also from observations in [7] it is seen that an increase in the turbulent energy results in an increase in the frictional dissipation. This would mean an increase in the input spectral density. Thus the decrease in H(S) is more or less compensated by an increase in $\eta_{in}(\vec{k})$, resulting in no net change in $\eta_o(\vec{k})$.

5. Prediction

On the basis of Kotelnikov's theorem, a random process which is singular (i.e. satisfying the condition that $\int_{0}^{\infty} \frac{I_n \eta_0 (\omega)}{\omega^2 + \omega_1^2} d\omega$ divergence ω_1 being a constant) can be extrapolated for a very long time. A band-limited function satisfying the condition

$$\eta_0 (\omega) = 0 \quad \text{for} \quad \omega < -\omega_0 \quad \text{and} \quad \omega > \omega_0$$

and being bounded, i.e.

$$\eta_0 (\omega) \leqslant \eta_1 \quad \text{for} \quad -\omega_0 < \omega < \omega_0 \tag{5.1}$$

can be assumed singular, within certain limitations. In the present model the upper limit of the function is set by the high frequency filter while the lower limit is set by the sensitivity of the measuring instruments.

The random velocity function can now be expressed in terms of its earlier values at times, $t - \tau, t - 2\tau, \ldots$ where $\tau > 0$ is a suitably chosen time interval. Then the pth difference of $u_0 (t)$ can be written as,

$$\Delta_p (t) = u_0(t) - \binom{p}{1} u_0 (t - \tau) + \binom{p}{2} u_0(t - 2\tau) \ldots$$

$$\ldots (-1)^p \binom{p}{p} u_0 (t - p\tau) \tag{5.2}$$

where $\binom{p}{q}$ is the binomial coefficient

$$\binom{p}{q} = \frac{p!}{q! \, (p - q)!} = \frac{p \, (p - 1) \ldots \ldots (p - q + 1)}{q!}$$

The random functions $u_0(t)$ and $\Delta_p(t)$ are connected by a linear transformation. If we represent $u_0(t)$ as,

$$u_0(t) \; = \; u_\omega \, e^{i\omega t} \tag{5.3}$$

u_ω being the complex amplitude, then at the output we would get a function

$$x(t) \; = \; x_\omega \, e^{i\omega t} \tag{5.4}$$

where $x_\omega = K(\omega) \, u_\omega$

$K(\omega)$ being the transfer function of the filter. Utilising this principle we get from equation (5.2).

$$\Delta_p(t) \; = \; (1 - e^{-i\omega\tau})^p \, u_0(t)$$

so that $K(\omega) = (1 - e^{-i\omega\tau})^p$

or $|K(\omega)|^2 = (2 \, \text{Sin} \, \frac{\omega\tau}{2})^{2p}$ $\hspace{2cm}$ (5.5)

It follows from equation (5.2) that

$$\overline{\Delta_p^2(t)} \; = \; \frac{1}{2\Pi} \int_{-\infty}^{\infty} [K(\omega)]^2 \; \phi_0(\omega) \, d\omega$$

$$\leqslant \; \frac{\eta_1}{2\Pi} \int_{-\infty}^{\infty} (2 \sin \frac{\omega\tau}{2})^{2p} \, d\omega \tag{5.6}$$

If we choose a value of τ, such that

$$2 \, \text{Sin} \, \frac{\omega_0 \tau}{2} \; < \; 1 \hspace{2cm} \text{i.e.} \hspace{1cm} \tau < \frac{\Pi}{3\,\omega_0}$$

then we have, the limiting value

$$\lim_{p \to \infty} \overline{\Delta_p^2 (t)} = 0$$

Equation (5.2) can now be rewritten as

$$u_0(t) = \binom{p}{1} u_0(t - \tau) - \binom{p}{2} u_0(t - 2\tau) + \dots$$

$$+ \dots - (-1)^p \binom{p}{p} u_0(t - p\tau) + \Delta_p(t) \qquad (5.7)$$

where $\Delta_p(t)$ can be neglected if p is large.

Thus we can find $u_0(t)$ from its past values $u_0(t - \tau)$, $u_0(t - 2\tau) \dots$ etc. Having found $u_0(t)$ we can use this value to predict $u_0(t + \tau)$, $u_0(t + 2\tau) \dots$ etc.

The main limitation in the method is that the boundaries of the band should be very sharp for the function to be really singular.

Acknowledgements

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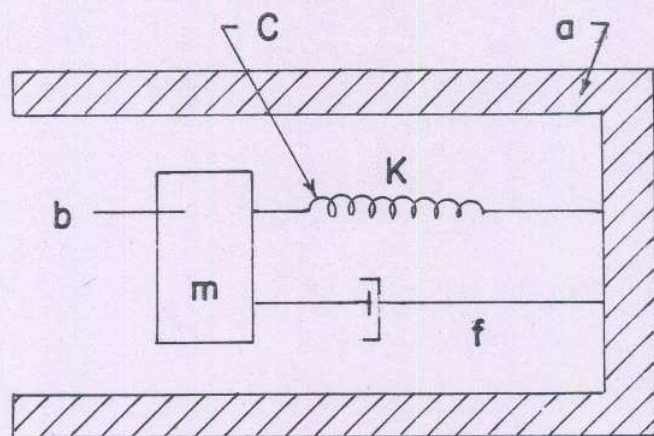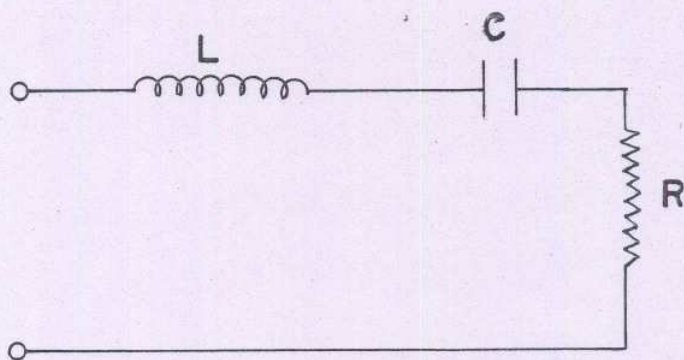   Met.Phy., C.S.I.R.O., Australia,
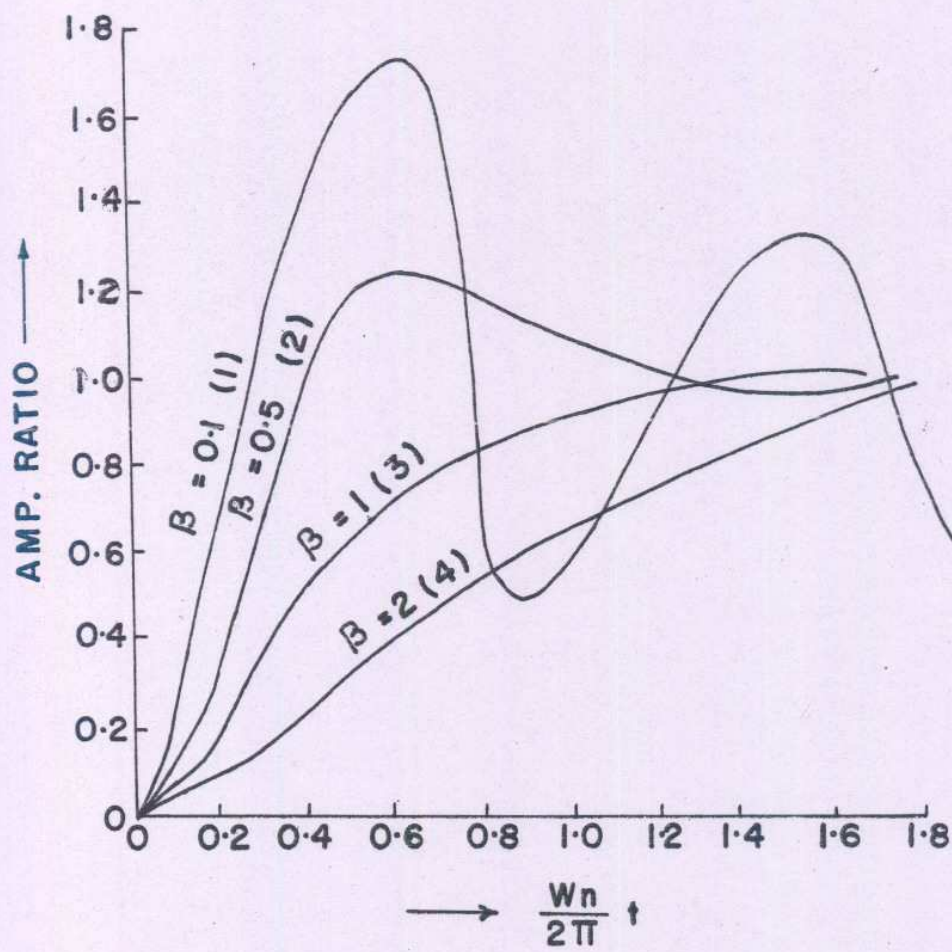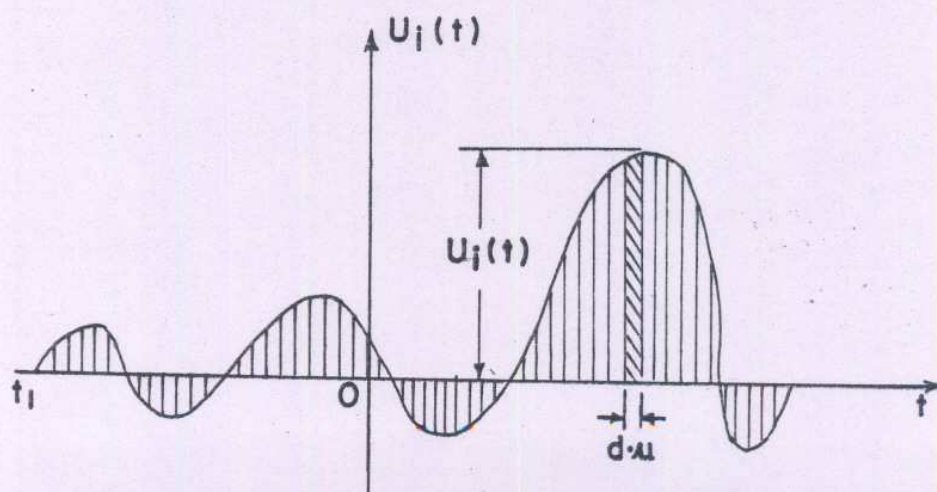   Tech.Paper No.5, pp.9-10

FIG. 1

FIG. 2

FIG. 3

FIG.4