

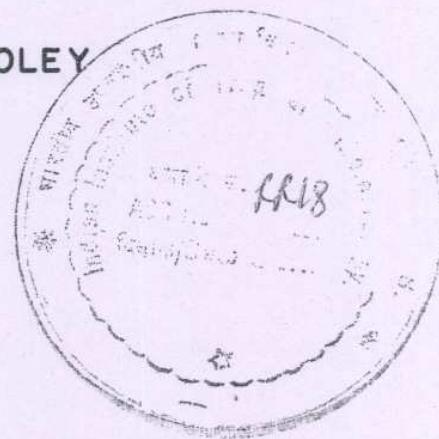
RESEARCH REPORT

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SUITABLE TABLES FOR APPLICATION
OF GAMMA PROBABILITY MODEL TO
RAINFALL

by

D. A. MOOLEY



INDIAN INSTITUTE OF TROPICAL METEOROLOGY

Ramdurg House
Ganeshkhind Road
Poona-5, India

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ABSTRACT

Gamma probability model has been found to be a good fit to daily depression rain and rain of time scale from 5 days to seasonal and annual rain over India. In view of the wide applicability of the model to Indian rainfall, simple and suitable tables have been given for fitting the model to rainfall and for obtaining the requisite probabilities, particularly for the benefit of those who have no access to the electronic computer. Before using these tables, one has to compute only the ratio, geometric mean/arithmetic mean of the individual non-zero rainfall amounts.

1. INTRODUCTION

Studies by Mooley and Crutcher (1968), and by Mooley (1970, 1971, 1973[a], 1973[b]) have shown that the two-parameter gamma probability distribution is a good fit to monthly and seasonal rainfall over the Asian summer monsoon area, to 5-day, seasonal and annual rainfall over the Indian subcontinent and to 24-hour depression rainfall in the different quadrants of monsoon depression over India.

Applying the tests for significance of g_1 and g_2 , which are Fisher's measures of skewness and kurtosis, and of the chi-square statistic, it is seen that while normal distribution shows a good fit to summer monsoon seasonal rainfall and annual rainfall at stations in some parts of India, it does not give a good fit to seasonal and annual rainfall at stations over the major portion of India. It has also been seen that the simple square root, cube root and logarithmic transformations are of limited utility in normalising the monthly rainfall in the field of the summer monsoon over Asia. Normal distribution is a special case of the gamma distribution. It is thus clear that gamma distribution probability model which covers a wide range of skewness and kurtosis can be applied to rainfall to obtain the rainfall probabilities required by a variety of user interests. Before applying the distribution to obtain the required rainfall probabilities, the parameters of the gamma distribution have to be estimated from the data by some suitable method. Fisher (1922) has shown that when the distribution is close to normal the moment estimates differ little from the Maximum Likelihood (M.L.) estimates but when the deviations from normality increase, the efficiency of the moment estimates falls off rather rapidly. It has been proved by him and others that M.L. estimates are consistent and that if sufficient estimates exist, they will be the M.L. estimates.

Fisher (1924) has shown that when inconsistent and inefficient estimates of the parameters are used, the computed Chi-square statistic measures not only the deviations of observations from the hypothesis but also the deviations due to errors in the estimation of the parameters. In view of these merits, Fisher has advised the use of M.L. estimates. The difficulty in obtaining the M.L. estimates is that the equation to be solved for the shape parameter of the gamma distribution is transcendental and the same cannot be solved except by iteration on an electronic computer. Again, the integral required to be evaluated to obtain the requisite rainfall probabilities cannot be evaluated except by numerical methods with the help of an electronic computer. These difficulties inhibit the use of this distribution. The purpose of the present paper, is to make available suitable and simple tables for obtaining the M.L. estimate of the shape parameter and for obtaining the requisite probabilities; so that those who do not have the access to the electronic computer, can apply the gamma distribution to rainfall and get the desired information on probabilities.

2. M.L. ESTIMATES OF THE PARAMETERS OF GAMMA DISTRIBUTION

The probability density function of the two-parameter gamma distribution is given by

$$f(X) = \frac{x^{\gamma-1} e^{-x/\beta}}{\beta^\gamma \Gamma(\gamma)}, \text{ for } X > 0; \gamma, \beta > 0 \dots [i]$$

$$= 0 \quad , \quad \text{for } X \leq 0$$

where γ and β are the shape and the scale parameters respectively of the distribution and $\Gamma(\gamma)$ is the gamma function of γ . The start of the distribution is at the origin. As $\gamma \rightarrow \infty$, gamma distribution \rightarrow normal distribution. Thus when γ is large, the distribution is either Gaussian or near - Gaussian.

The equations for M.L. estimates $(\hat{\gamma}, \hat{\beta})$ of the parameters of the gamma distribution are given by

$$\Psi(\hat{\gamma}) - \ln(\hat{\gamma}) - \ln(R) = 0 \dots [ii]$$

$$\hat{\beta} = \bar{X} / \hat{\gamma} \dots [iii]$$

where \bar{X} is the Arithmetic Mean (A.M.) and R , the ratio of geometric mean (G.M.) to arithmetic mean of x_1, x_2, \dots, x_n , the non-zero values of the variable X and $\Psi(\hat{\gamma})$, the di-gamma function, $= \frac{\partial \ln[\Gamma(\hat{\gamma})]}{\partial \hat{\gamma}}$ and

\ln is the natural logarithm. It can be seen that equation [ii] is transcendental and that for different values of R , the value of $\hat{\gamma}$ can be computed to any desired accuracy by using an electronic computer. The value of $\hat{\gamma}$ is uniquely determined by R .

Now, since $GM \leq AM$ always, R ($=$ geometric mean/arithmetic mean) ≤ 1 . Taking into account the fact that rainfall observations can be accurate upto 1 cent or 0.3 mm only and the fact that often the rainfall record is in whole millimeters, the ratio R is not expected to be more accurate than unity in the third place of decimal. Values of this ratio for different time scales of rainfall are generally found to lie between .100 and .990, which values correspond to $\hat{\gamma} = 0.3$ and 50.0 respectively. Thus the values of $\hat{\gamma}$ tabulated for $R = .100$ to $.990$ at intervals of .001 will be adequate for all practical purposes and there would be no need of interpolation. These values are tabulated under Table 1. Value of $\hat{\gamma}$ is obtained by reading the value of R in Table 1. Values of $\hat{\gamma}$ have been calculated to a much higher accuracy and these are rounded to the fourth decimal place and then put in Table 1. From the value of $\hat{\gamma}$, $\hat{\beta}$ can be obtained by using equation [iii].

3. TEST FOR GAMMA DISTRIBUTION

Where previous studies by research workers indicate that gamma distribution would be a good fit, the distribution may be applied to the data and requisite probabilities may be calculated as shown in the next section. Otherwise,

it would be advisable to test the fit of the gamma distribution to the data under consideration. Chi-square test, Kolmogorov - Smirnov test and the variance ratio test can be applied to test the fit. Of these, the variance ratio (V.R) test which is simple and easy may be applied and other tests may be applied if felt necessary. The V.R. test has been suggested by Cochran (1954) for Poisson and Binomial distributions. In general, it can be used for all distributions for which the theoretical variance can be obtained independently from the parameters estimated by a method other than the method of moments. If $\hat{\beta}$ and $\hat{\gamma}$ are M.L. estimates of β and γ respectively, then the theoretical variance is $\hat{\beta}^2\hat{\gamma}$. The test statistic is

$$X_{ij}^2 = \frac{\sum_{i=1}^n (x_i - \bar{x})^2}{\hat{\beta}^2\hat{\gamma}},$$

n being the number of occasions of non-zero rainfall and \bar{x} , the arithmetic mean of non-zero rainfall amounts.

This is to be referred to Chi-square tables with $(n - 1)$ degrees of freedom. Fisher and Yates (1957) have pointed out that for $(n - 1)$ degrees of freedom, the expression $\sqrt{(2 X_{ij}^2)}$ is approximately normally distributed

with mean $\sqrt{(2n - 3)}$ and standard deviation unity, provided $(n - 1)$ is greater than 30. To test the significance of X^2 , based on more than 30 degrees of freedom, we calculate the expression, $\sqrt{(2 X^2)} - \sqrt{(2n - 3)}$, which is a normal deviate with zero mean and unit standard deviation and see if this expression is significant at the adopted level of significance. If $(n - 1) \leq 30$, reference may be made to Chi-square tables to see if X^2 is significant.

4. RAINFALL PROBABILITIES

The probability p_1 of non-zero rain being less than x_1 is given by

$$p_1 = \int_0^{x_1} \frac{x^{\gamma-1} e^{-x/\beta}}{\beta^\gamma \Gamma(\gamma)} dx \dots [iv]$$

If P is the probability of zero rain as obtained from rainfall data, then p'_1 , the probability of rain being less than x_1 is given by

$$p'_1 = P + (1 - P) p_1 \dots [v]$$

If the rainfall distribution includes zero rain and gamma distribution is a good fit to non-zero rain then the rainfall distribution is referred to by Thom (1951) as mixed gamma distribution.

The requirements in respect of rainfall probability vary substantially from user to user. Hence instead of tabulating probabilities of rainfall being less than specified rainfall amounts it is preferable to tabulate specific percentiles of the gamma distribution for which probability density function is given by

$$f(z) = \frac{z^{\hat{\gamma}-1} e^{-z}}{\Gamma(\hat{\gamma})} \dots \quad [vi]$$

$$\text{where } z = x/\beta$$

The following equation has been solved on the computer by iteration process for z when $P_z = .05$ to $.95$ at interval of $.05$, utilising the procedure given in detail by Mooley (1973(a))

$$P_z = \int_0^z \frac{z^{\hat{\gamma}-1} e^{-z}}{\Gamma(\hat{\gamma})} dz \dots \quad [vii]$$

Table 2 gives the 5th to 95th percentiles of the distribution given by equation (vi), at intervals of 5, for $\hat{\gamma} = 0.3(.1) 5.0(.2) 10.0(.5) 30.0(1.0) 50.0$, i.e. from 0.3 to 5.0 at interval of .1, etc. The tables for $\hat{\gamma} > 50.0$ are not needed, since in this situation Gaussian distribution is invariably a good fit to rainfall. Values of the percentiles have been computed to a much higher accuracy and these are rounded to the fourth decimal place

before inclusion in Table 2. From these tabulated percentiles, any user can interpolate and obtain the rainfall probabilities required by him.

Gamma and di-gamma functions are required in the computation of Tables 1 and 2. Appendix I indicates how these functions are computed. Appendix I also indicates how the integral in equation [iv] is evaluated.

The use of Tables 1 and 2 is illustrated with examples in Appendix II.

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APPENDIX I

(1) Computation of Gamma Function

The most well-known form in which the gamma function has been defined is the one given by Euler.

$$\Gamma(t) = \int_0^\infty x^{t-1} e^{-x} dx \quad \dots \quad [AI.1]$$

This function has been approximated by the following logarithmic form of the Sterling series (Davis, 1933).

$$\begin{aligned} \ln[\Gamma(t)] &\sim [(t-0.5)\ln(t)-t + 0.5\ln(2\pi) \\ &+ \sum_{m=1}^{\infty} (-1)^{m-1} B_m / \{2m(2m-1)t^{2m-1}\}] \\ &\dots \quad [AI.2] \end{aligned}$$

where B_m stands for Bernoulli numbers and \ln , for natural logarithm.

Taking the terms upto and including the one containing B_3 in equation AI.2, we have,

$$\begin{aligned} \ln(\Gamma(t)) &\sim [(t-0.5)\ln(t)-t + 0.5\ln(2\pi) + \frac{B_1}{2t} - \frac{B_2}{12t^3} \\ &+ \frac{B_3}{30t^5} - \frac{B_4}{56t^7} + \frac{B_5}{90t^9} - \frac{B_6}{132t^{11}} + \frac{B_7}{182t^{13}} - \frac{B_8}{240t^{15}}] \\ &\dots \quad \dots \quad [AI.3] \end{aligned}$$

The Bernoulli numbers are given by

$$B_1 = 1/6, \quad B_2 = 1/30, \quad B_3 = 1/42, \quad B_4 = 1/30,$$

$$B_5 = 5/66, \quad B_6 = 691/2730, \quad B_7 = 7/6, \quad B_8 = 3617/510$$

$$\begin{aligned}\log_{10} [\Gamma(t)] &= \log_{10}(e) \times \ln([\Gamma(t)]) \\ &= .434294481903252 \times \ln[\Gamma(t)]\end{aligned}$$

Utilising equation [AI.3], we get,

$\log_{10}[\Gamma(4)] = .778151250380501$, which value is less by 4 in the twelfth decimal place than the value given by Davis (1933). Thus for $t \geq 4$, the value of the gamma function obtained by using equation [AI.3] is correct to 11 places of decimals. When $t < 4$, a value of the same order of accuracy can be obtained by using the following reduction formula

$$\begin{aligned}\log_{10} [\Gamma(t)] &= [\log_{10}(e)] [\ln(\Gamma(t+4) - \ln(t+3) \\ &\quad - \ln(t+2) - \ln(t+1) - \ln(t)] \quad [AI.4]\end{aligned}$$

since $\Gamma(t+4) = (t+3)(t+2)(t+1)t \Gamma(t)$, by property of the gamma function. $\ln[\Gamma(t+4)]$ is obtained by using equation [AI.3].

(2) Computation of di-gamma function

$$\text{di-gamma function} = \Psi(t) = \frac{\partial \ln[\Gamma(t)]}{\partial t}$$

Differentiating equation [AI.3], we get,

$$\begin{aligned}\Psi(t) \sim & [\ln(t) - \frac{1}{2t} - \frac{B_1}{2t^2} + \frac{B_2}{4t^4} - \frac{B_3}{6t^6} + \frac{B_4}{8t^8} \\ & - \frac{B_5}{10t^{10}} + \frac{B_6}{12t^{12}} - \frac{B_7}{14t^{14}} + \frac{B_8}{16t^{16}}] \dots \quad [AI.5]\end{aligned}$$

At $t = 50.5$, equation [AI.5] gives

$\Psi(50.5) = 3.912039668366257$, which is smaller by 3 in the ninth decimal place than that given in tables by Davis (1933).

At $t = 80.5$, the error in $\Psi(t)$ is about 4 in the tenth decimal place. Thus the gain in accuracy from $t = 50.5$ to $t = 80.5$ is relatively small. Hence the following reduction formula was applied when t was less than 50.

$$\Psi(t) = \Psi(t + 50) - \left(\frac{1}{t+49} + \frac{1}{t+48} + \dots + \frac{1}{t} \right) \dots [AI.6]$$

$\Psi(t + 50)$ is computed by using equation [AI.5]

(3) Computation of tri-gamma function

$$\text{tri-gamma function} = \Psi'(t) = \frac{\partial^2 \ln[\Gamma(t)]}{\partial t^2}$$

Differentiating [AI.5], we get,

$$\begin{aligned} \Psi'(t) \sim & \left[\frac{1}{t} + \frac{1}{2t^2} + \frac{B_1}{t^3} - \frac{B_2}{t^5} + \frac{B_3}{t^7} - \frac{B_4}{t^9} \right. \\ & \left. + \frac{B_5}{t^{11}} - \frac{B_6}{t^{13}} + \frac{B_7}{t^{15}} - \frac{B_8}{t^{17}} \right] [AI.7] \end{aligned}$$

At $t = 6$, (AI.7) gives,

$\Psi'(6) = .1813229558$, which is larger by unity in the tenth decimal place than that given in tables by Davis (1933).

Thus when $t \geq 6$, [AI.7] may be used to obtain $\Psi'(t)$, whereas, when $t < 6$, the following reduction formula may be used.

$$\Psi'(t) = \Psi'(t+6) + \left[\frac{1}{(t+5)^2} + \dots + \frac{1}{(t+1)^2} + \frac{1}{t^2} \right] \dots [AI.8]$$

(4) Evaluation of the probability integral

$$P_X = \int_0^X \frac{x^{\gamma-1} e^{-x/\beta}}{\beta^\gamma \Gamma(\gamma)} dx$$

Integrating successively by parts, we get,

$$P_X = \frac{e^{-x/\beta} (x/\beta)^{\gamma}}{\Gamma(\gamma+1)} \left[1 + \frac{(x/\beta)}{(\gamma+1)} + \frac{(x/\beta)^2}{(\gamma+1)(\gamma+2)} + \dots + \frac{(x/\beta)^n}{(\gamma+1)(\gamma+2)\dots(\gamma+n)} + \dots \right] [AI.9]$$

By taking a sufficiently large number of terms of the series within the bracket, P_X can be evaluated to any desired accuracy.

APPENDIX II

Example 1 : Ahmedabad rainfall for the pentad number 39
July
(10-14₁) for the period 1901-1960.

Number of occasions of zero rain = 7

Empirical probability of zero rain = 7/60 = .117

Arthmetic mean of non-zero pentad rain = 5.77 cm

Geometric mean of non-zero pentad rain = 2.44 cm

R, the ratio, geometric mean/arithmetic mean = .423

Entering this value of R in table 2, we get $\hat{\gamma}$,

the M.L. estimate of the shape parameter of the gamma distribution fitted to the pentad rainfall = .7021

$\hat{\beta}$, the M.L. estimate of the scale parameter =

$\frac{\text{arithmetic mean}}{\hat{\gamma}} = 8.219 \text{ cm}$

Suppose we require the probability of the pentad rainfall not exceeding 6.92 cm

Now $\frac{6.92}{\hat{\beta}} (= Z) = .842$

From Table 2, corresponding to $\hat{\gamma} = 0.702$, and $Z = 0.842$, p, the probability of non-zero rain being less than 6.92 cm can be interpolated as 70.96 percent ≈ 71 percent or .71.

p', the probability of rain $< 6.92 \text{ cm}$
 $= .117 + (1.0 - .117) \times .71$, utilising equation (v)
 $= .744$

If we need probability of rain exceeding 6.92 cms,
the same is $1.0 - .744$ i.e. .256.

Example 2 : Nagpur monthly rainfall for September
for the period 1854-1960.

Suppose we want to obtain the monthly rainfall amount which will not be exceeded with a probability of .70. What we want is the 70th percentile of the gamma distribution fitted to September rainfall of Nagpur.

Number of occasions of no rain = zero

Arithmetic mean of the monthly rain = 19.28 cm

Geometric mean of the monthly rain = 16.25 cm

R, the ratio, geometric mean/arithmetic mean = .843

Entering this value of R in Table 1,

We get the value of $\hat{\gamma}$ = 3.0842

$$\text{Hence } \hat{\beta} = \frac{19.28}{3.0842} = 6.25$$

From Table 2, 70th percentile for $\hat{\gamma}$ = 3.0 is 3.6156

" " " " $\hat{\gamma}$ = 3.1 is 3.7313

Interpolating, we get 70th percentile for Z-distribution (where $Z = X/\hat{\beta}$) for $\hat{\gamma} = 3.0842$

$$= 3.7130$$

Multiplying this by the scale parameter, we get the 70th percentile of the gamma distribution fitted to September rainfall of Nagpur.

$$= 3.7130 \times 6.25 \text{ cm}$$

$$= 23.21 \text{ cm.}$$

Table 1 : Shape parameter, $\hat{\gamma}$, for different values of R ($R = \text{the ratio, Geometric mean of rainfall/ arithmetic mean of rainfall}$)

R	0	1	2	3	4	5	6	7	8	9
.10	2995	3007	3018	3029	3040	3051	3062	3073	3084	3095
.11	3106	3117	3128	3139	3150	3161	3172	3183	3194	3205
.12	3216	3226	3237	3248	3259	3270	3281	3291	3302	3313
.13	3324	3335	3345	3356	3367	3377	3388	3399	3410	3420
.14	3431	3442	3453	3463	3474	3485	3495	3506	3517	3527
.15	3538	3549	3560	3570	3581	3592	3602	3613	3624	3634
.16	3645	3656	3667	3677	3688	3699	3709	3720	3731	3741
.17	3752	3763	3774	3784	3795	3806	3817	3827	3838	3849
.18	3860	3870	3881	3892	3903	3913	3924	3935	3946	3957
.19	3967	3978	3989	4000	4011	4022	4033	4043	4054	4065
.20	4076	4087	4098	4109	4120	4131	4142	4153	4164	4175
.21	4186	4197	4208	4219	4230	4241	4252	4263	4274	4285
.22	4296	4307	4319	4330	4341	4352	4363	4375	4386	4397
.23	4406	4419	4431	4442	4453	4465	4476	4487	4499	4510
.24	4521	4533	4544	4556	4567	4579	4590	4602	4613	4625
.25	4636	4648	4659	4671	4683	4694	4706	4718	4729	4741
.26	4753	4764	4776	4788	4800	4812	4823	4835	4847	4859
.27	4871	4883	4895	4907	4919	4931	4943	4955	4967	4979
.28	4991	5003	5015	5026	5040	5052	5064	5077	5089	5101
.29	5113	5126	5138	5151	5163	5175	5188	5200	5213	5225
.30	5238	5251	5263	5276	5288	5301	5314	5327	5339	5352
.31	5365	5378	5391	5403	5416	5429	5443	5455	5468	5481
.32	5494	5507	5521	5534	5547	5560	5573	5587	5600	5613
.33	5627	5640	5653	5667	5680	5694	5707	5721	5735	5748
.34	5762	5776	5789	5803	5817	5831	5844	5858	5872	5886
.35	5900	5914	5928	5942	5956	5970	5985	5999	6013	6027
.36	6042	6056	6070	6085	6099	6114	6128	6143	6157	6172
.37	6187	6201	6216	6231	6245	6260	6275	6290	6305	6320
.38	6335	6350	6365	6381	6396	6411	6426	6441	6457	6472
.39	6488	6503	6519	6534	6550	6565	6581	6597	6613	6628
.40	6644	6660	6676	6692	6708	6724	6740	6756	6773	6789
.41	6805	6821	6838	6854	6871	6887	6904	6920	6937	6954
.42	6971	6987	7004	7021	7038	7055	7072	7089	7106	7124
.43	7141	7158	7175	7193	7210	7228	7245	7263	7281	7299
.44	7316	7334	7352	7370	7388	7406	7424	7442	7461	7479
.45	7497	7516	7534	7553	7571	7590	7609	7627	7646	7665
.46	7684	7703	7722	7741	7760	7779	7797	7818	7838	7857
.47	7877	7896	7915	7936	7956	7975	7994	8015	8036	8056
.48	8076	8096	8117	8137	8158	8178	8199	8219	8240	8261
.49	8282	8303	8324	8345	8367	8388	8410	8431	8452	8474
.50	8496	8517	8539	8561	8583	8605	8627	8650	8672	8695
.51	8717	8740	8762	8785	8808	8831	8854	8877	8900	8923
.52	8947	8970	8993	9017	9041	9065	9089	9113	9137	9161
.53	9185	9209	9234	9258	9283	9308	9333	9358	9383	9408
.54	9433	9458	9484	9509	9535	9561	9587	9613	9639	9665
.55	9691	9717	9744	9771	9797	9824	9851	9878	9905	9932
.56	9960	9987	1.0015	0043	0071	0099	0127	0155	0183	0212
.57	1.0240	0269	0298	0327	0356	0385	0414	0444	0473	0503
.58	1.0533	0563	0593	0623	0654	0684	0715	0746	0777	0808
.59	1.0839	0871	0902	0934	0966	0997	1030	1062	1094	1127
.60	1.1160	1193	1226	1259	1292	1326	1359	1393	1427	1461
.61	1.1496	1530	1565	1599	1635	1670	1705	1741	1776	1812
.62	1.1848	1885	1921	1958	1994	2031	2069	2106	2143	2181
.63	1.2219	2257	2296	2334	2373	2412	2451	2490	2530	2569
.64	1.2609	2649	2690	2731	2773	2813	2854	2895	2937	2979
.65	1.3021	3064	3106	3149	3192	3236	3279	3323	3387	3411
.66	1.3456	3501	3546	3591	3637	3683	3729	3775	3822	3869
.67	1.3916	3964	4012	4060	4109	4157	4206	4255	4304	4354
.68	1.4404	4455	4505	4556	4608	4659	4711	4764	4816	4869
.69	1.4923	4976	5030	5084	5139	5194	5249	5305	5361	5417
.70	1.5474	5531	5589	5647	5705	5764	5823	5882	5942	6002

R	0	1	2	3	4	5	6	7	8	9
.71	1.6063	6124	6185	6247	6310	6372	6435	6499	6563	6628
.72	1.6693	6758	6824	6890	6957	7024	7092	7160	7229	7298
.73	1.7368	7438	7509	7580	7652	7724	7797	7870	7944	8019
.74	1.8093	8169	8245	8322	8399	8477	8556	8635	8715	8795
.75	1.8876	8958	9040	9123	9207	9291	9376	9462	9548	9635
.76	1.9723	9811	9901	9991	2.0081	0173	0265	0358	0452	0546
.77	2.0642	0738	0835	0933	1032	1131	1232	1333	1436	1539
.78	2.1643	1748	1854	1961	2069	2178	2288	2399	2511	2624
.79	2.2738	2853	2970	3087	3206	3325	3446	3568	3691	3816
.80	2.3941	4068	4197	4326	4457	4589	4722	4857	4993	5131
.81	2.5270	5410	5552	5696	5841	5987	6135	6285	6437	6590
.82	2.6744	6901	7059	7219	7380	7544	7709	7877	8041	8217
.83	2.8391	8566	8743	8923	9104	9288	9474	9662	9853	3.0046
.84	3.0241	0439	0639	0842	1047	1255	1466	1679	1895	2114
.85	3.2336	2561	2789	3020	3254	3492	3732	3976	4224	4475
.86	3.4729	4987	5249	5515	5784	6058	6335	6617	6903	7193
.87	3.7488	7787	8091	8400	8714	9032	9356	9685	4.0019	0359
.88	4.0705	1056	1413	1776	2146	2522	2905	3294	3690	4093
.89	4.4504	4922	5348	5782	6224	6674	7133	7601	8078	8564
.90	4.9061	9567	5.0083	0610	1148	1697	2259	2832	3417	4015
.91	5.4628	5257	5893	6547	7220	7982	8603	9322	6.0057	0811
.92	6.1584	2376	3181	4022	4878	5757	6658	7585	8537	9516
.93	7.0523	1560	2620	3725	4857	6023	7226	8467	9749	8.1072
.94	8.2469	3852	5318	6828	8395	9.0020	1705	3453	5268	7154

R	0	1	2	3	4
	5	6	7	8	9
.95	9.9116 11.0232	10.1158 11.2759	10.3285 11.5403	10.5502 11.8172	10.7816 12.1077
.96	12.4127 14.1989	12.7333 14.6191	13.0707 15.0649	13.4265 15.5385	13.8019 16.0426
.97	16.5803 19.9142	17.1551 20.7476	17.7710 21.6535	18.4325 22.6418	19.1449 23.7241
.98	24.9147 33.2485	26.2306 35.6296	27.6921 38.3769	29.3268 41.5822	31.1851 45.3701
.99	49.9157				

Note : For R = .100 to .949, the integral part, if any, of \hat{Y} , is given where it occurs first, as well as in the column under the heading zero; elsewhere, the four figures without the decimal sign give the first four decimal places.

Table 2 : Specified percentiles of gamma distribution function
 $(P_x = \int_0^x e^{-z} z^{\gamma-1} dz)$

$\frac{Y}{\sqrt{x}}$	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2
.05	.3211x10 ⁻⁴	.4147x10 ⁻³	.1966x10 ⁻²	.5645x10 ⁻²	.1216x10 ⁻¹	.2190x10 ⁻¹	.3496x10 ⁻¹	.5129x10 ⁻¹	.7075x10 ⁻¹	.9315x10 ⁻¹
.10	.3237x10 ⁻³	.2349x10 ⁻²	.7895x10 ⁻²	.1806x10 ⁻¹	.3315x10 ⁻¹	.5298x10 ⁻¹	.7720x10 ⁻¹	.1054	.1371	.1719
.15	.1252x10 ⁻²	.6492x10 ⁻²	.1768x10 ⁻¹	.3589x10 ⁻¹	.6009x10 ⁻¹	.8974x10 ⁻¹	.1241	.1625	.2044	.2494
.20	.3270x10 ⁻²	.1339x10 ⁻¹	.3209x10 ⁻¹	.5880x10 ⁻¹	.9234x10 ⁻¹	.1315	.1754	.2231	.2742	.3280
.25	.6900x10 ⁻²	.2356x10 ⁻¹	.5077x10 ⁻¹	.8677x10 ⁻¹	.1298	.1783	.2312	.2877	.3471	.4090
.30	.1273x10 ⁻¹	.3754x10 ⁻¹	.7424x10 ⁻¹	.1200	.1725	.2302	.2919	.3567	.4241	.4936
.35	.2142x10 ⁻¹	.5591x10 ⁻¹	.1029	.1588	.2210	.2877	.3579	.4308	.5058	.5826
.40	.3374x10 ⁻¹	.7936x10 ⁻¹	.1375	.2038	.2757	.3514	.4300	.5108	.5933	.6771
.45	.5061x10 ⁻¹	.1087	.1787	.2558	.3375	.4223	.5092	.5978	.6876	.7784
.50	.7313x10 ⁻¹	.1451	.2275	.3157	.4074	.5014	.5967	.6931	.7903	.8879
.55	.1027	.1898	.2853	.3851	.4871	.5903	.6942	.7985	.9030	1.0076
.60	.1413	.2448	.3542	.4659	.5785	.6913	.8039	.9163	1.0283	1.1400
.65	.1913	.3126	.4367	.5610	.6847	.8074	.9291	1.0498	1.1697	1.2887
.70	.2566	.3973	.5371	.6747	.8101	.9432	1.0744	1.2040	1.3320	1.4587
.75	.3429	.5048	.6617	.8137	.9615	1.1059	1.2473	1.3863	1.5231	1.6581
.80	.4601	.6456	.8212	.9890	1.1506	1.3074	1.4601	1.6094	1.7560	1.9001
.85	.6266	.8391	1.0361	1.2220	1.3994	1.5702	1.7358	1.8971	2.0548	2.2094
.90	.8848	1.1298	1.3528	1.5605	1.7571	1.9453	2.1267	2.3026	2.4740	2.6415
.95	1.3723	1.6620	1.9207	2.1590	2.3826	2.5951	2.7990	2.9957	3.1867	3.3727
	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0	2.1	2.2
.05	.1183	.1459	.1759	.2081	.2422	.2783	.3160	.3554	.3962	.4385
.10	.2096	.2497	.2922	.3367	.3831	.4311	.4808	.5318	.5842	.6378
.15	.2970	.3465	.3989	.4527	.5082	.5652	.6236	.6832	.7440	.8058
.20	.3842	.4425	.5026	.5643	.6275	.6920	.7577	.8244	.8921	.9607
.25	.4730	.5389	.6063	.6750	.7451	.8162	.8883	.9613	1.0351	1.1097
.30	.5649	.6377	.7118	.7871	.8634	.9406	1.0186	1.0973	1.1768	1.2568
.35	.6608	.7403	.8208	.9022	.9844	1.0674	1.1509	1.2350	1.3197	1.4048
.40	.7621	.8479	.9346	1.0219	1.1098	1.1982	1.2871	1.3764	1.4661	1.5561
.45	.8699	.9621	1.0547	1.1478	1.2413	1.3351	1.4292	1.5235	1.6180	1.7128
.50	.9860	1.0844	1.1830	1.2818	1.3808	1.4799	1.5791	1.6783	1.7778	1.8771
.55	1.1123	1.2169	1.3215	1.4260	1.5305	1.6349	1.7393	1.8436	1.9478	2.0519
.60	1.2514	1.3624	1.4731	1.5835	1.6936	1.8034	1.9130	2.0223	2.1314	2.2403
.65	1.4070	1.5246	1.6416	1.7580	1.8739	1.9893	2.1043	2.2188	2.3330	2.4469
.70	1.5843	1.7068	1.8324	1.9552	2.0772	2.1985	2.3191	2.4392	2.5588	2.6778
.75	1.7915	1.9235	2.0542	2.1837	2.3123	2.4399	2.5666	2.6926	2.8179	2.9425
.80	2.0421	2.1823	2.3208	2.4579	2.5937	2.7283	2.8618	2.9943	3.1259	3.2567
.85	2.3613	2.5109	2.6585	2.8043	2.9484	3.0911	3.2324	3.3724	3.5114	3.6493
.90	2.8056	2.9670	3.1257	3.2822	3.4366	3.5893	3.7403	3.8897	4.0378	4.1846
.95	3.5544	3.7325	3.9074	4.0793	4.2487	4.4158	4.5808	4.7439	4.9052	5.0648

	2.3	2.4	2.5	2.6	2.7	2.8	2.9	3.0	3.1	3.2
.05	.4820	.5268	.5727	.6198	.6678	.7169	.7668	.8177	.8694	.9218
.10	.6925	.7484	.8052	.8629	.9215	.9809	1.0411	1.1021	1.1637	1.2260
.15	.8686	.9323	.9969	1.0623	1.1283	1.1951	1.2626	1.3306	1.3993	1.4685
.20	1.0301	1.1003	1.1713	1.2429	1.3151	1.3878	1.4612	1.5350	1.6094	1.6842
.25	1.1849	1.2608	1.3373	1.4143	1.4919	1.5699	1.6464	1.7273	1.8066	1.8863
.30	1.3373	1.4184	1.5000	1.5819	1.6643	1.7471	1.8303	1.9138	1.9976	2.0817
.35	1.4903	1.5763	1.6626	1.7492	1.8361	1.9234	2.0109	2.0986	2.1866	2.2749
.40	1.6464	1.7369	1.8277	1.9188	2.0101	2.1016	2.1932	2.2851	2.3771	2.4693
.45	1.8077	1.9027	1.9980	2.0933	2.1888	2.2844	2.3801	2.4759	2.5718	2.6678
.50	1.9766	2.0762	2.1757	2.2753	2.3750	2.4746	2.5743	2.6741	2.7738	2.8736
.55	2.1560	2.2600	2.3639	2.4677	2.5715	2.6753	2.7790	2.8826	2.9862	3.0897
.60	2.3491	2.4576	2.5659	2.6741	2.7822	2.8900	2.9978	3.1054	3.2129	3.3202
.65	2.5604	2.6736	2.7865	2.8992	3.0116	3.1237	3.2357	3.3474	3.4589	3.5702
.70	2.7964	2.9145	3.0322	3.1496	3.2666	3.3832	3.4995	3.6156	3.7313	3.8468
.75	3.0665	3.1899	3.3128	3.4352	3.5572	3.6787	3.7998	3.9204	4.0407	4.1607
.80	3.3867	3.5160	3.6446	3.7726	3.9100	4.0269	4.1532	4.2790	4.4043	4.5293
.85	3.7863	3.9224	4.0576	4.1921	4.3258	4.4588	4.5913	4.7231	4.8543	4.9849
.90	4.3302	4.4747	4.6182	4.7607	4.9023	5.0431	5.1831	5.3223	5.4608	5.5987
.95	5.2230	5.3798	5.5352	5.6895	5.8426	5.9947	6.1457	6.2958	6.4450	6.5933
	3.3	3.4	3.5	3.6	3.7	3.8	3.9	4.0	4.1	4.2
.05	.9751	1.0290	1.0837	1.1390	1.1949	1.2515	1.3086	1.3663	1.4246	1.4833
.10	1.2889	1.3525	1.4166	1.4812	1.5454	1.6120	1.6782	1.7448	1.8118	1.8793
.15	1.5382	1.6084	1.6791	1.7503	1.8219	1.8939	1.9663	2.0391	2.1123	2.1858
.20	1.7594	1.8351	1.9112	1.9876	2.0644	2.1415	2.2190	2.2968	2.3749	2.4533
.25	1.9663	2.0467	2.1274	2.2084	2.2898	2.3713	2.4532	2.5353	2.6177	2.7003
.30	2.1660	2.2507	2.3357	2.4208	2.5062	2.5918	2.6777	2.7637	2.8499	2.9364
.35	2.3633	2.4520	2.5408	2.6299	2.7191	2.8084	2.8980	2.9876	3.0775	3.1674
.40	2.5616	2.6540	2.7466	2.8393	2.9322	3.0251	3.1182	3.2113	3.3046	3.3979
.45	2.7639	2.8600	2.9563	3.0525	3.1489	3.2453	3.3418	3.4383	3.5349	3.6315
.50	2.9733	3.0731	3.1729	3.2727	3.3725	3.4724	3.5722	3.6721	3.7719	3.8718
.55	3.1932	3.2966	3.4000	3.5033	3.6066	3.7099	3.8131	3.9163	4.0194	4.1225
.60	3.4275	3.5346	3.6416	3.7485	3.8553	3.9621	4.0687	4.1753	4.2817	4.3881
.65	3.6813	3.7923	3.9031	4.0137	4.1241	4.2345	4.3446	4.4547	4.5646	4.6744
.70	3.9620	4.0770	4.1917	4.3062	4.4205	4.5346	4.6485	4.7622	4.8758	4.9891
.75	4.2803	4.3996	4.5186	4.6373	4.7557	4.8739	4.9918	5.1094	5.2269	5.3441
.80	4.6538	4.7779	4.9016	5.0250	5.1480	5.2707	5.3930	5.5150	5.6368	5.7583
.85	5.1151	5.2448	5.3739	5.5027	5.6310	5.7589	5.8864	6.0135	6.1403	6.2667
.90	5.7359	5.8725	6.0085	6.1440	6.2789	6.4134	6.5473	6.6808	6.8138	6.9464
.95	6.7408	6.8876	7.0336	7.1789	7.3235	7.4675	7.6109	7.7537	7.8959	8.0375

	4.3	4.4	4.5	4.6	4.7	4.8	4.9	5.0	5.2	5.4
.05	1.5426	1.6023	1.6626	1.7232	1.7843	1.8459	1.9078	1.9731	2.0960	2.2232
.10	1.9472	2.0154	2.0841	2.1531	2.2225	2.2922	2.3622	2.4326	2.5742	2.7171
.15	2.2596	2.3338	2.4083	2.4830	2.5581	2.6335	2.7091	2.7850	2.9376	3.0911
.20	2.5319	2.6108	2.6900	2.7695	2.8491	2.9291	3.0092	3.0895	3.2509	3.4130
.25	2.7831	2.8662	2.9494	3.0329	3.1165	3.2004	3.2844	3.3686	3.3375	3.7071
.30	3.0230	3.1097	3.1967	3.2837	3.3710	3.4584	3.5459	3.6336	3.8094	3.9856
.35	3.2575	3.3478	3.4381	3.5286	3.6192	3.7099	3.8007	3.8916	4.0737	4.2562
.40	3.4914	3.5849	3.6785	3.7722	3.8660	3.9598	4.0537	4.1477	4.3359	4.5243
.45	3.7282	3.8249	3.9217	4.0185	4.1154	4.2123	4.3092	4.4062	4.6002	4.7944
.50	3.9717	4.0715	4.1714	4.2713	4.3712	4.4711	4.5710	4.6709	4.8707	5.0706
.55	4.2256	4.3286	4.4316	4.5345	4.6375	4.7404	4.8433	4.9461	5.1517	5.3572
.60	4.4944	4.6007	4.7068	4.8129	4.9189	5.0249	5.1308	5.2366	5.4481	5.6594
.65	4.7840	4.8936	5.0030	5.1123	5.2215	5.3306	5.4397	5.5486	5.7661	5.9833
.70	5.1023	5.2153	5.3282	5.4409	5.5535	5.6659	5.7782	5.8904	6.1143	6.3378
.75	5.4610	5.5778	5.6944	5.8108	5.9269	6.0429	6.1588	6.2744	6.5053	6.7355
.80	5.8795	6.0004	6.1211	6.2415	6.3617	6.4817	6.6014	6.7210	6.9594	7.1971
.85	6.3928	6.5186	6.6440	6.7692	6.8940	7.0186	7.1429	7.2670	7.5143	7.7607
.90	7.0786	7.2104	7.3418	7.4729	7.6036	7.7339	7.8639	7.9936	8.2520	8.5093
.95	8.1787	8.3193	8.4595	8.5992	8.7384	8.8772	9.0156	9.1535	9.4283	9.7015
	5.6	5.8	6.0	6.2	6.4	6.6	6.8	7.0	7.2	7.4
.05	2.3519	2.4818	2.6130	2.7454	2.8788	3.0133	3.1488	3.2853	3.4227	3.5610
.10	2.8610	3.0059	3.1519	3.2988	3.4465	3.5952	3.7446	3.8948	4.0457	4.1973
.15	3.2455	3.4008	3.5569	3.7138	3.8714	4.0296	4.1886	4.3481	4.5083	4.6691
.20	3.5759	3.7394	3.9037	4.0685	4.2340	4.4000	4.5666	4.7337	4.9012	5.0693
.25	3.8772	4.0479	4.2192	4.3910	4.5632	4.7359	4.9091	5.0827	5.2566	5.4310
.30	4.1623	4.3395	4.5171	4.6951	4.8735	5.0522	5.2313	5.4107	5.5905	5.7705
.35	4.4391	4.6223	4.8058	4.9895	5.1736	5.3580	5.5425	5.7274	5.9124	6.0977
.40	4.7130	4.9019	5.0910	5.2803	5.4698	5.6594	5.8493	6.0392	6.2294	6.4197
.45	4.9887	5.1831	5.3776	5.5723	5.7670	5.9615	6.1567	6.3517	6.5468	6.7419
.50	5.2704	5.4703	5.6702	5.8700	6.0699	6.2698	6.4697	6.6696	6.8696	7.0695
.55	5.5626	5.7679	5.9732	6.1783	6.3834	6.5883	6.7932	6.9981	7.2028	7.4075
.60	5.8704	6.0813	6.2919	6.5024	6.7127	6.9228	7.1328	7.3426	7.5523	7.7619
.65	6.2002	6.4168	6.6330	6.8490	7.0648	7.2802	7.4955	7.7105	7.9252	8.1398
.70	6.5608	6.7834	7.0055	7.2273	7.4488	7.6699	7.8906	8.1110	8.3312	8.5510
.75	6.9651	7.1942	7.4227	7.6508	7.8783	8.1055	8.3322	8.5585	8.7844	9.0099
.80	7.4341	7.6704	7.9060	8.1410	8.3754	8.6093	8.8426	9.0754	9.3077	9.5396
.85	8.0062	8.2508	8.4947	8.7377	8.9801	9.2217	9.4627	9.7031	9.9429	10.1821
.90	8.7654	9.0206	9.2747	9.5279	9.7801	10.0316	10.2822	10.5321	10.7812	11.0296
.95	9.9733	10.2438	10.5130	10.7811	11.0479	11.3138	11.5786	11.8424	12.1053	12.3672

	7.6	7.8	8.0	8.2	8.4	8.6	8.8	9.0	9.2	9.4
.05	3.7001	3.8401	3.9808	4.1223	4.2645	4.4074	4.5510	4.6952	4.8401	4.9846
.10	4.3496	4.5026	4.6561	4.8103	4.9650	5.1203	5.2761	5.4325	5.5893	5.7466
.15	4.8304	4.9922	5.1545	5.3173	5.4806	5.6444	5.8085	5.9731	6.1381	6.3035
.20	5.2378	5.4067	5.5761	5.7458	5.9159	6.0864	6.2573	6.4285	6.6030	6.7719
.25	5.6057	5.7807	5.9561	6.1318	6.3078	6.4842	6.6608	6.8376	7.0145	7.1922
.30	5.9508	6.1313	6.3122	6.4933	6.6746	6.8561	7.0379	7.2199	7.4021	7.5846
.35	6.2832	6.4689	6.6548	6.8409	7.0271	7.2136	7.4002	7.5869	7.7738	7.9609
.40	6.6101	6.8007	6.9914	7.1822	7.3731	7.5642	7.7553	7.9466	8.1380	8.3294
.45	6.9371	7.1324	7.3277	7.5232	7.7186	7.9142	8.1098	8.3054	8.5011	8.6968
.50	7.2694	7.4693	7.6692	7.8692	8.0691	8.2691	8.4690	8.6690	8.8689	9.0689
.55	7.6121	7.8167	8.0212	8.2257	8.4301	8.6345	8.8388	9.0430	9.2472	9.4514
.60	7.9713	8.1806	8.3898	8.5988	8.8078	9.0166	9.2253	9.4340	9.6425	9.8539
.65	8.3542	8.5683	8.7823	8.9961	9.2097	9.4232	9.6365	9.8497	10.0627	10.2755
.70	8.7706	8.9899	9.2089	9.4277	9.6463	9.8647	10.0828	10.3007	10.5184	10.7359
.75	9.2351	9.4599	9.6844	9.9086	10.1325	10.3561	10.5794	10.8024	11.0252	11.2477
.80	9.7710	10.0020	10.2325	10.4627	10.6925	10.9220	11.1510	11.3798	11.6082	11.8363
.85	10.4208	10.6589	10.8965	11.1337	11.3703	11.6066	11.8424	12.0777	12.3127	12.5473
.90	11.2774	11.5244	11.7709	12.0168	12.2621	12.5058	12.7510	12.9947	13.2379	13.4806
.95	12.6283	12.8886	13.1481	13.4068	13.6648	13.9221	14.1787	14.4346	14.6900	14.9446
	9.6	9.8	10.0	10.5	11.0	11.5	12.0	12.5	13.0	13.5
.05	5.1316	5.2782	5.4254	5.7957	6.1690	6.5453	6.9242	7.3057	7.6896	8.0757
.10	5.9044	6.0626	6.2213	6.6198	7.0207	7.4240	7.8293	8.2367	8.6459	9.0569
.15	6.4693	6.6354	6.8019	7.2197	7.6394	8.0610	8.4843	8.9092	9.3357	9.7636
.20	6.9440	7.1165	7.2892	7.7223	8.1570	8.5933	9.0309	9.4699	9.9101	10.3515
.25	7.3699	7.5478	7.7259	8.1722	8.6198	9.0686	9.5186	9.9697	10.4217	10.8747
.30	7.7672	7.9500	8.1329	8.5911	9.0504	9.5105	9.9716	10.4335	10.8962	11.3596
.35	8.1481	8.3354	8.5229	8.9921	9.4621	9.9328	10.4042	10.8762	11.3487	11.8218
.40	8.5210	8.7127	8.9044	9.3842	9.8644	10.3451	10.8262	11.3078	11.7897	12.2720
.45	8.8926	9.0885	9.2843	9.7742	10.2644	10.7548	11.2454	11.7362	12.2272	12.7184
.50	9.2688	9.4688	9.6687	10.1686	10.6685	11.1684	11.6684	12.1683	12.6682	13.1682
.55	9.6555	9.8596	10.0636	10.5735	11.0832	11.5926	12.1018	12.6109	13.1198	13.6265
.60	10.0592	10.2675	10.4757	11.0957	11.5153	12.0345	12.5532	13.0715	13.5894	14.1070
.65	10.4882	10.7008	10.9132	11.4438	11.9736	12.5028	13.0313	13.5592	14.0865	14.6133
.70	10.9532	11.1703	11.3873	11.9289	12.4695	13.0092	13.5480	14.0860	14.6232	15.1596
.75	11.4700	11.6920	11.9138	12.4674	13.0196	13.5707	14.1206	14.6694	15.2173	15.7642
.80	12.0641	12.2915	12.5188	13.0856	13.6507	14.2144	14.7767	15.3376	15.8973	16.4558
.85	12.7815	13.0153	13.2488	13.8310	14.4112	14.9896	15.5662	16.1412	16.7147	17.2868
.90	13.7228	13.9646	14.2060	14.8075	15.4066	16.0034	16.5981	17.1908	17.7816	18.3706
.95	15.1987	15.4523	15.7052	16.3353	16.9622	17.5862	18.2075	18.8262	19.4426	20.0566

	14.0	14.5	15.0	15.5	16.0	16.5	17.0	17.5	18.0	18.5
.05	8.4639	8.8542	9.2463	9.6433	10.0360	10.4333	10.8321	11.2325	11.6343	12.0375
.10	9.4696	9.8839	10.2996	10.7168	11.1353	11.5551	11.9761	12.3983	12.8215	13.2460
.15	10.1929	10.6234	11.0552	11.4881	11.9221	12.3572	12.7932	13.2302	13.6681	14.1069
.20	10.7940	11.2375	11.6621	12.1275	12.5739	13.0211	13.4691	13.9179	14.3675	14.8177
.25	11.3286	11.7833	12.2388	12.6951	13.1521	13.6097	14.0680	14.5270	14.9865	15.4466
.30	11.8237	12.2885	12.7539	13.2199	13.6864	14.1535	14.6210	15.0891	15.5576	16.0266
.35	12.2954	12.7695	13.2441	13.7191	14.1944	14.6702	15.1464	15.6229	16.0998	16.5769
.40	12.7546	13.2376	13.7208	14.2043	14.6881	15.1722	15.6565	16.1411	16.6258	17.1108
.45	13.2097	13.7012	14.1929	14.6847	15.1767	15.6687	16.1609	16.6532	17.1457	17.6382
.50	13.6681	14.1681	14.6680	15.1680	15.6679	16.1679	16.6679	17.1678	17.6678	18.1678
.55	14.1370	14.6454	15.1536	15.6617	16.1697	16.6776	17.1853	17.6929	18.2004	18.7078
.60	14.6243	15.1413	15.6579	16.1743	16.6904	17.2063	17.7219	18.2373	18.7525	19.2674
.65	15.1396	15.6654	16.1908	16.7157	17.2402	17.7644	18.2881	18.8116	19.3346	19.8574
.70	15.6954	16.2306	16.7651	17.2991	17.8325	18.3653	18.8977	19.4296	19.9610	20.4920
.75	16.3102	16.8555	17.3999	17.9435	18.4865	19.0288	19.5704	20.1114	20.6518	21.1917
.80	17.0133	17.5697	18.1251	18.6796	19.2332	19.7859	20.3378	20.8890	21.4394	21.9891
.85	17.8575	18.4269	18.9951	19.5622	20.1281	20.6931	21.2570	21.8200	22.3820	22.9432
.90	18.9580	19.5437	20.1280	20.7109	21.2924	21.8726	22.4516	23.0294	23.6061	24.1817
.95	20.6686	21.2785	21.8865	22.4927	23.0971	23.6999	24.3012	24.9009	25.4992	26.0962
	19.0	19.5	20.0	20.5	21.0	21.5	22.0	22.5	23.0	23.5
.05	12.4420	12.8477	13.2547	13.6628	14.0720	14.4824	14.8937	15.3061	15.7195	16.1338
.10	13.6715	14.0979	14.5253	14.9535	15.3827	15.8127	16.2436	16.6752	17.1076	17.5407
.15	14.5465	14.9870	15.4282	15.8701	16.3128	16.7561	17.2001	17.6448	18.0500	18.5359
.20	15.2687	15.7203	16.1725	16.6253	17.0787	17.5327	17.9872	18.4422	18.8977	19.3538
.25	15.9073	16.3585	16.8301	17.2923	17.7550	18.2160	18.6816	19.1455	19.6098	20.0746
.30	16.4960	16.9658	17.4360	17.9055	18.3775	18.8488	19.3204	19.7924	20.2646	20.7372
.35	17.0544	17.5322	18.0103	18.4887	18.9673	19.4462	19.9254	20.4047	20.8844	21.3642
.40	17.5960	18.0814	18.5670	19.0527	19.5387	20.0248	20.5111	20.9975	21.4841	21.9708
.45	18.1308	18.6236	19.1164	19.6093	20.1024	20.5955	21.0886	21.5819	22.0752	22.5686
.50	18.6677	19.1677	19.5677	20.1676	20.6676	21.1676	21.6676	22.1676	22.6675	23.1675
.55	19.2151	19.7223	20.2294	20.7365	21.2434	21.7503	22.2570	22.7637	23.2704	23.7769
.60	19.7822	20.2967	20.8111	21.3253	21.8393	22.3531	22.8668	23.3803	23.8937	24.4069
.65	20.3798	20.9020	21.4239	21.9454	22.4668	22.9878	23.5086	24.0292	24.5496	25.0697
.70	21.0225	21.5527	22.0824	22.6118	23.1408	23.6695	24.1978	24.7259	25.2536	25.7809
.75	21.7310	22.2697	22.8080	23.3458	23.8831	24.4200	24.9564	25.4925	26.0281	26.5633
.80	22.5381	23.0865	23.6343	24.1814	24.7280	25.2740	25.8195	26.3644	26.9088	27.4528
.85	23.5036	24.0631	24.6219	25.1800	25.7373	26.2939	26.8499	27.4052	27.9560	28.5141
.90	24.7563	25.3299	25.9025	26.4743	27.0451	27.6151	28.1843	28.7526	29.3203	29.8871
.95	26.6918	27.2861	27.8792	28.4712	29.0620	29.6518	30.2404	30.8281	31.4148	32.0006

	24.00	24.5	25.0	25.5	26.0	26.5	27.0	27.5	28.0	28.5
.05	16.5490	16.9652	17.3821	17.7999	18.2185	18.6379	19.0581	19.4790	19.9006	20.3230
.10	17.9746	18.4091	18.8443	19.2802	19.7167	20.1538	20.5915	21.0298	21.4687	21.9081
.15	18.9824	19.4294	19.8770	20.3251	20.7737	21.2228	21.6724	22.1224	22.5729	23.0239
.20	19.8103	20.2572	20.7246	21.1824	21.6407	22.0993	22.5584	23.0178	23.4776	23.9378
.25	20.5397	21.0052	21.4710	21.9372	22.4038	22.8706	23.3378	23.8052	24.2730	24.7411
.30	21.2101	21.6832	22.1567	22.6304	23.1043	23.5785	24.0530	24.5277	25.0026	25.4778
.35	21.8443	22.3245	22.8050	23.2857	23.7665	24.2476	24.7288	25.2102	25.6918	26.1735
.40	22.8577	22.9447	23.4319	23.9192	24.4066	24.8941	25.3818	25.8695	26.3574	26.8454
.45	23.0621	23.5557	24.0493	24.5430	25.0367	25.5305	26.0244	26.5183	27.0123	27.5064
.50	23.6675	24.1575	24.6675	25.1675	25.6674	26.1674	26.6674	27.1674	27.6674	28.1674
.55	24.2834	24.7898	25.2961	25.8024	26.3087	26.8148	27.3209	27.8270	28.3330	28.8389
.60	24.9200	25.4330	25.9458	26.4585	26.9710	27.4835	27.9958	28.5080	29.0201	29.5321
.65	25.5896	26.1093	26.6288	27.1481	27.6672	28.1861	28.7049	29.2235	29.7419	30.2601
.70	26.3080	26.8349	27.3614	27.8877	28.4137	28.9394	29.4650	29.9902	30.5153	31.0401
.75	27.0982	27.6327	28.1668	28.7006	29.2340	29.7672	30.3000	30.8325	31.3647	31.8966
.80	27.9963	28.5393	29.0819	29.6241	30.1658	30.7071	31.2481	31.7886	32.3288	32.8686
.85	29.0676	29.6206	30.1730	30.7249	31.2763	31.8272	32.3776	32.9275	33.4770	34.0260
.90	30.4533	31.0188	31.5836	32.1477	32.7112	33.2741	33.8364	34.3981	34.9593	35.5199
.95	32.5854	33.1693	33.7524	34.3346	34.9161	35.4967	36.0766	36.6557	37.2342	37.8119
	29.0	29.5	30.0	31.0	32.0	33.0	34.0	35.0	36.0	37.0
.05	20.7460	21.1697	21.5940	22.4445	23.2975	24.1527	25.0101	25.8696	26.7312	27.5946
.10	22.3480	22.7885	23.2294	24.1129	24.9981	25.8852	26.7740	27.6645	28.5565	29.4500
.15	23.4753	23.9271	24.3793	25.2850	26.1922	27.1008	28.0108	28.9221	29.8348	30.7486
.20	24.3.83	24.8591	25.3203	26.2437	27.1662	28.0940	29.0209	29.9489	30.8780	31.8079
.25	25.2094	25.6760	26.1469	27.0855	28.0250	28.9655	29.9069	30.8492	31.7923	32.7361
.30	25.9532	26.4288	26.9046	27.8566	28.8098	29.7635	30.7179	31.6730	32.6288	33.5851
.35	26.6554	27.1375	27.6197	28.5846	29.5500	30.5160	31.4825	32.4495	33.4170	34.3849
.40	27.3335	27.8217	28.3100	29.2669	30.2642	31.2418	32.2198	33.1981	34.1767	35.1556
.45	28.0004	28.4946	28.9888	29.9773	30.9660	31.9549	32.9440	33.9332	34.9226	35.9121
.50	28.6674	29.1673	29.6673	30.6673	31.6673	32.6673	33.6673	34.6672	35.6672	36.6672
.55	29.3448	29.8506	30.3564	31.3676	32.3791	33.3901	34.4011	35.4118	36.4224	37.4328
.60	30.0440	30.5557	31.0671	32.0905	33.1132	34.1355	35.1575	36.1792	37.2005	38.2216
.65	30.7782	31.2961	31.8139	32.8489	33.8835	34.9174	35.9509	36.9839	38.0164	39.0484
.70	31.5647	32.0891	32.6133	33.6613	34.7060	35.7542	36.7998	37.8446	38.8889	39.9325
.75	32.4283	32.9596	33.4907	34.5521	35.6126	36.6720	37.7306	38.7863	39.8452	41.1402
.80	33.4081	33.9472	34.4860	35.5627	36.6380	37.7122	38.7653	39.8573	40.9283	41.9983
.85	34.5745	35.1228	35.6705	36.7649	37.8577	38.9491	40.0390	41.1277	42.2150	43.3012
.90	35.3795	36.6395	37.1085	38.3151	39.4298	40.5427	41.6540	42.7635	43.8715	44.9780
.95	36.3889	37.9653	39.5410	40.6905	41.8376	42.9825	44.1251	45.2656	46.4041	47.5407

	38.0	39.0	40.0	41.0	42.0	43.0	44.0	45.0
.05	28.4599	29.3270	30.1957	31.0661	31.9381	32.8116	33.6866	34.5630
.10	30.3449	31.2413	32.1389	33.0379	33.9380	34.8394	35.7419	36.6455
.15	31.6636	32.5797	33.4969	34.4151	35.3344	36.2546	37.1757	38.0977
.20	32.7389	33.6707	34.6035	35.5371	36.4714	37.4066	38.3426	39.2792
.25	33.6808	34.6262	35.5723	36.5190	37.4664	38.4145	39.3631	40.3123
.30	34.5421	35.4990	36.4577	37.4163	38.3754	39.3350	40.2950	41.2555
.35	35.3533	36.3221	37.2913	38.2608	39.2308	40.2011	41.1717	42.1427
.40	36.1348	37.1142	38.0940	39.0740	40.0542	41.0347	42.0153	42.9963
.45	36.9018	37.8916	38.8815	39.8716	40.8618	41.8521	42.8425	43.8330
.50	37.6672	38.6672	39.6672	40.6672	41.6671	42.6672	43.6671	44.6671
.55	38.4431	39.4533	40.4633	41.4732	42.4830	43.4927	44.5022	45.5117
.60	39.2424	40.2629	41.2831	42.3031	43.3229	44.3424	45.3616	46.3807
.65	40.0800	41.1112	42.1420	43.1724	44.2024	45.2321	46.2614	47.2904
.70	40.9755	42.0179	43.0599	44.1012	45.1421	46.1825	47.2224	48.2619
.75	41.9566	43.0112	44.0651	45.1184	46.1709	47.2229	48.2742	49.3250
.80	43.0673	44.1354	45.2027	46.2691	47.3347	48.3995	49.4635	50.5269
.85	44.3662	45.4701	46.5529	47.6346	48.7154	49.7952	50.8741	51.9520
.90	46.0831	47.1868	48.2891	49.3902	50.4810	51.5886	52.6861	53.7825
.95	48.6755	49.8085	50.9397	52.0694	53.1974	54.3239	55.4490	56.5726

	46.0	47.0	48.0	49.0	50.0
.05	35.4408	36.3199	37.2003	38.0819	38.9647
.10	37.5502	38.4560	39.3627	40.2704	41.1791
.15	39.0206	39.9443	40.8689	41.7942	42.7203
.20	40.2166	41.1546	42.0933	43.0327	43.9727
.25	41.2621	42.2125	43.1633	44.1147	45.0666
.30	42.2165	43.1779	44.1397	45.1019	46.0645
.35	43.1140	44.0856	45.0576	46.0298	47.0023
.40	43.9774	44.9587	45.9403	46.9220	47.9039
.45	44.8237	45.8144	46.8053	47.7962	48.7872
.50	45.6671	46.6671	47.6671	48.6671	49.6671
.55	46.5210	47.5303	48.5394	49.5485	50.5574
.60	47.3996	48.4182	49.4367	50.4549	51.4730
.65	48.3191	49.3475	50.3755	51.4033	52.4308
.70	49.3009	50.3395	51.3777	52.4155	53.4529
.75	50.3752	51.4248	52.4739	53.5225	54.5706
.80	51.5895	52.6514	53.7127	54.7733	55.8336
.85	53.0291	54.1054	55.1808	56.2555	57.3294
.90	54.8778	55.9721	57.0654	58.1576	59.2490
.95	57.6949	58.8158	59.9355	61.0539	62.1711